ON C^*-ALGEBRAS ASSOCIATED WITH C^*-CORRESPONDENCES

TAKESHI KATSURA

Abstract. We study C^*-algebras arising from C^*-correspondences, which was introduced by the author. We prove the gauge-invariant uniqueness theorem, and obtain conditions for our C^*-algebras to be nuclear, exact, or satisfy the Universal Coefficient Theorem. We also obtain a 6-term exact sequence of K-groups involving the K-groups of our C^*-algebras.

0. Introduction

In [Ka2], we introduce a method to construct C^*-algebras from C^*-correspondences. This construction is similar to the one of Cuntz-Pimsner algebras [P], and in fact these two constructions coincide when the left action of a given C^*-correspondence is injective. However, when the left action of a C^*-correspondence is not injective, our construction differs from the one in [P]. Our construction of C^*-algebras from C^*-correspondences whose left actions are not injective is motivated by the constructions of graph algebras of graphs with sinks in [FLR], C^*-algebras from topological graphs in [Ka1], and crossed products by Hilbert C^*-bimodules in [AEE]. In fact, our construction generalizes all of these constructions. In our next paper [Ka3], we will explain that our C^*-algebras have a nice property which crossed products by automorphisms also have.

In this paper, we prove several theorems on our C^*-algebras, which generalize or improve known results on Cuntz-Pimsner algebras or other classes of C^*-algebras. After preliminaries of C^*-correspondences and their representations in Sections 1 and 2, we give definitions of our C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ for a C^*-correspondence X in Section 3. Sections 4 and 5 are preparatory sections for our main theorems. In Section 4, we review constructions of Fock spaces and Fock representations. Most of the results in this section have been already known. In Section 5, we analyze so-called cores. Main theorems can be found in Sections 6, 7 and 8. In Section 6, we prove the gauge-invariant uniqueness theorems of our C^*-algebras, which will play an important role in the analysis of their ideals in [Ka3]. In Section 7, we give necessary and sufficient conditions for our C^*-algebras to be nuclear or exact. In Section 8, we give a 6-term exact sequence of K-groups which seems to be helpful to compute K-groups of our C^*-algebras. We also give a sufficient condition for our C^*-algebras to satisfy the Universal Coefficient Theorem of [RS].

1991 *Mathematics Subject Classification.* Primary 46L05,46L55.

Key words and phrases. Hilbert modules, C^*-correspondences, Cuntz-Pimsner algebras, gauge action, nuclear, exact, K-groups.

The author was supported in part by a Research Fellowship for Young Scientists of the Japan Society for the Promotion of Science.

The author would like to thank Narutaka Ozawa who kindly gave Example 7.7 and the proof of Proposition A.13. He is also grateful to N. Christopher Phillips for many comments, and to Yasuyuki Kawahigashi for constant encouragement. This paper was written while the author was staying at University of Oregon. He would like to thank people there for their warm hospitality.

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers, and by $\mathbb{T}$ the group consisting of complex numbers whose absolute values are 1. We use a convention that $\gamma(A, B) = \{\gamma(a, b) \in D \mid a \in A, b \in B\}$ for a map $\gamma\colon A \times B \to D$ such as inner products, multiplications or representations. We denote by $\overline{\operatorname{span}}\{\cdots\}$ the closure of linear spans of $\{\cdots\}$. An ideal of a C^*-algebra means a closed two-sided ideal.

1. C^*-correspondences

We use [L2] for the general reference of Hilbert C^*-modules and C^*-correspondences.

Definition 1.1. Let A be a C^*-algebra. A (right) *Hilbert A-module* X is a Banach space with a right action of the C^*-algebra A and an A-valued inner product $\langle\cdot,\cdot\rangle_X\colon X \times X \to A$ satisfying certain conditions.

Recall that a Hilbert A-module X is said to be *full* if $\overline{\operatorname{span}}\langle X, X\rangle_X = A$. We do not assume that Hilbert C^*-modules X are full. For a C^*-algebra A, A itself is a Hilbert A-module where the inner product is defined by $\langle\xi, \eta\rangle_A = \xi^*\eta$, and the right action is multiplication.

Definition 1.2. For Hilbert A-modules X, Y, we denote by $\mathcal{L}(X, Y)$ the space of all adjointable operators from X to Y. For $\xi \in X$ and $\eta \in Y$, the operator $\theta_{\eta,\xi} \in \mathcal{L}(X, Y)$ is defined by $\theta_{\eta,\xi}(\zeta) = \eta\langle\xi, \zeta\rangle_X \in Y$ for $\zeta \in X$. We define $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ by

$$\mathcal{K}(X, Y) = \overline{\operatorname{span}}\{\theta_{\eta,\xi} \in \mathcal{L}(X, Y) \mid \xi \in X, \eta \in Y\}.$$

For a Hilbert A-module X, we set $\mathcal{L}(X) = \mathcal{L}(X, X)$, which is a C^*-algebra, and $\mathcal{K}(X) = \mathcal{K}(X, X)$, which is an ideal of $\mathcal{L}(X)$.

Definition 1.3. For a C^*-algebra A, we say that X is a *C^*-correspondence* over A when X is a Hilbert A-module and a $*$-homomorphism $\varphi_X\colon A \to \mathcal{L}(X)$ is given.

We refer to φ_X as the left action of a C^*-correspondence X. A C^*-correspondence X over A is said to be *non-degenerate* if $\overline{\operatorname{span}}(\varphi_X(A)X) = X$. We do not assume that C^*-correspondences are non-degenerate.

Let A be a C^*-algebra. We can define a left action of the C^*-algebra A on the Hilbert A-module A by the multiplication. Thus we get a C^*-correspondence over A, which is called the *identity correspondence* over A and denoted by A. Note that the left action φ_A of the identity correspondence A gives an isomorphism from A onto $\mathcal{K}(A) \subset \mathcal{L}(X)$.

Definition 1.4. Let X, Y be C^*-correspondences over a C^*-algebra A. We denote by $X \odot Y$ the quotient of the algebraic tensor product of X and Y by the subspace generated by $(\xi a) \otimes \eta - \xi \otimes (\varphi_Y(a)\eta)$ for $\xi \in X$, $\eta \in Y$ and $a \in A$. We can define

an A-valued inner product, right and left actions of A on $X \odot Y$ by

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle_{X \otimes Y} = \big\langle \eta, \varphi_Y(\langle \xi, \xi' \rangle_X)\eta' \big\rangle_Y$$
$$(\xi \otimes \eta)a = \xi \otimes (\eta a), \quad \varphi_{X \otimes Y}(a)(\xi \otimes \eta) = (\varphi_X(a)\xi) \otimes \eta,$$

for $\xi, \xi' \in X$, $\eta, \eta' \in Y$ and $a \in A$. One can show that these operations are well-defined and extend to the completion of $X \odot Y$ with respect to the norm coming from the A-valued inner product defined above (see [L2, Proposition 4.5]). Thus the completion of $X \odot Y$ is a C^*-correspondence over A. This C^*-correspondence is called the *tensor product* of X and Y, and denoted by $X \otimes Y$.

By definition, we have

$$X \otimes Y = \overline{\operatorname{span}}\{\xi \otimes \eta \mid \xi \in X, \eta \in Y\},$$

and $(\xi a) \otimes \eta = \xi \otimes (\varphi_Y(a)\eta)$ for $\xi \in X$, $\eta \in Y$ and $a \in A$.

Definition 1.5. For a C^*-correspondence X over a C^*-algebra A and $n \in \mathbb{N}$, we define a C^*-correspondence $X^{\otimes n}$ over A by $X^{\otimes 0} = A$, $X^{\otimes 1} = X$, and $X^{\otimes (n+1)} = X \otimes X^{\otimes n}$ for $n \geq 1$.

For each $n \in \mathbb{N}$, the left action $\varphi_{X^{\otimes n}}$ of the C^*-correspondence $X^{\otimes n}$ will be simply denoted by $\varphi_n \colon A \to \mathcal{L}(X^{\otimes n})$. For a positive integer n, we have

$$X^{\otimes n} = \overline{\operatorname{span}}\{\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \mid \xi_1, \xi_2, \dots, \xi_n \in X\}.$$

Note that for positive integers n, m, there exists a natural isomorphism between $X^{\otimes n} \otimes X^{\otimes m}$ and $X^{\otimes (n+m)}$. We have such isomorphisms for $m = 0$, but for $n = 0$ we just get an injection $X^{\otimes 0} \otimes X^{\otimes m} \to X^{\otimes m}$. When X is non-degenerate, this injection is actually an isomorphism, but it is not surjective in general.

Definition 1.6. Let n be a positive integer, and take $S \in \mathcal{L}(X^{\otimes n})$. For each $m \in \mathbb{N}$, we define $S \otimes \mathrm{id}_m \in \mathcal{L}(X^{\otimes (n+m)})$ by $(S \otimes \mathrm{id}_m)(\xi \otimes \eta) = S(\xi) \otimes \eta$ for $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$.

We note that $S \otimes \mathrm{id}_0 = S$. The $*$-homomorphism $\mathcal{L}(X^{\otimes n}) \ni S \mapsto S \otimes \mathrm{id}_m \in \mathcal{L}(X^{\otimes (n+m)})$ is injective when φ_X is injective, but this is not the case in general. When X is non-degenerate, we can define $S \otimes \mathrm{id}_n \in \mathcal{L}(X^{\otimes n})$ for $S \in \mathcal{L}(X^{\otimes 0})$ and $n \geq 1$ because $X^{\otimes 0} \otimes X^{\otimes n} \cong X^{\otimes n}$. In this case, we have $a \otimes \mathrm{id}_n = \varphi_n(a)$ for $a \in A \cong \mathcal{K}(X^{\otimes 0})$. By abuse of notation, for $a \in A \cong \mathcal{K}(X^{\otimes 0})$ we use the notation $a \otimes \mathrm{id}_n$ for denoting $\varphi_n(a) \in \mathcal{L}(X^{\otimes n})$ even though X is degenerate. Note that we cannot define $S \otimes \mathrm{id}_n \in \mathcal{L}(X^{\otimes n})$ for $S \in \mathcal{L}(X^{\otimes 0})$ in general. In other words, the $*$-homomorphism $\varphi_n \colon A \to \mathcal{L}(X^{\otimes n})$ need not extend to a $*$-homomorphism $\mathcal{M}(A) \to \mathcal{L}(X^{\otimes n})$ unless X is non-degenerate.

Definition 1.7. Let us take $\xi \in X^{\otimes n}$ with $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, we define an operator $\tau_m^n(\xi) \in \mathcal{L}(X^{\otimes m}, X^{\otimes (n+m)})$ by

$$\tau_m^n(\xi) \colon X^{\otimes m} \ni \eta \mapsto \xi \otimes \eta \in X^{\otimes (n+m)}.$$

Note that for $a \in A = X^{\otimes 0}$, we have $\tau_m^0(a) = \varphi_m(a) \in \mathcal{L}(X^{\otimes m})$ for each $m \in \mathbb{N}$. Note also that $\tau_0^n \colon X^{\otimes n} \to \mathcal{L}(X^{\otimes 0}, X^{\otimes n})$ is an isometry onto $\mathcal{K}(X^{\otimes 0}, X^{\otimes n})$ for each $n \in \mathbb{N}$. The adjoint $\tau_m^n(\xi)^* \in \mathcal{L}(X^{\otimes (n+m)}, X^{\otimes m})$ of $\tau_m^n(\xi)$ satisfies that $\tau_m^n(\xi)^*(\zeta \otimes \eta) = \varphi_m(\langle \xi, \zeta \rangle_{X^{\otimes n}})\eta$ for $\zeta \in X^{\otimes n}$, $\eta \in X^{\otimes m}$. It is not difficult to see the following two lemmas.

Lemma 1.8. *For $n_1, n_2, m \in \mathbb{N}$ and $\xi_1 \in X^{\otimes n_1}, \xi_2 \in X^{\otimes n_2}$, we have*

$$\tau_{n_2+m}^{n_1}(\xi_1)\tau_m^{n_2}(\xi_2) = \tau_m^{n_1+n_2}(\xi_1 \otimes \xi_2) \qquad \text{in } \mathcal{L}(X^{\otimes m}, X^{\otimes(n_1+n_2+m)}).$$

Lemma 1.9. *For $n, m \in \mathbb{N}$, $\xi, \eta \in X^{\otimes n}$ and $a \in A$, we have the following;*

(i) $\tau_m^n(\xi)\tau_m^n(\eta)^* = \theta_{\xi,\eta} \otimes \mathrm{id}_m$ *in* $\mathcal{L}(X^{\otimes(n+m)})$,

(ii) $\tau_m^n(\xi)^*\tau_m^n(\eta) = \varphi_m(\langle \xi, \eta\rangle_{X^{\otimes n}})$ *in* $\mathcal{L}(X^{\otimes m})$,

(iii) $\tau_m^n(\xi)\varphi_m(a) = \tau_m^n(\xi a)$ *in* $\mathcal{L}(X^{\otimes m}, X^{\otimes(n+m)})$,

(iv) $\varphi_{n+m}(a)\tau_m^n(\xi) = \tau_m^n(\varphi_n(a)\xi)$ *in* $\mathcal{L}(X^{\otimes m}, X^{\otimes(n+m)})$.

2. Representations of C^*-correspondences

Definition 2.1. A *representation* of a C^*-correspondence X over A on a C^*-algebra B is a pair consisting of a $*$-homomorphism $\pi\colon A \to B$ and a linear map $t\colon X \to B$ satisfying

(i) $t(\xi)^*t(\eta) = \pi\big(\langle \xi, \eta\rangle_X\big)$ for $\xi, \eta \in X$,

(ii) $\pi(a)t(\xi) = t\big(\varphi_X(a)\xi\big)$ for $a \in A$, $\xi \in X$.

We denote by $C^*(\pi, t)$ the C^*-algebra generated by the images of π and t in B.

A representation of a C^*-correspondence was called an isometric covariant representation in [MS]. Note that for a representation (π, t) of X, we have $t(\xi)\pi(a) = t(\xi a)$ automatically because the condition (i) above, combining with the fact that π is a $*$-homomorphism, implies

$$\big\|t(\xi)\pi(a) - t(\xi a)\big\|^2 = \big\|\big(t(\xi)\pi(a) - t(\xi a)\big)^*\big(t(\xi)\pi(a) - t(\xi a)\big)\big\| = 0.$$

Note also that for $\xi \in X$, we have $\|t(\xi)\| \leq \|\xi\|_X$ because

$$\|t(\xi)\|^2 = \|t(\xi)^*t(\xi)\| = \|\pi(\langle \xi, \xi\rangle_X)\| \leq \|\langle \xi, \xi\rangle_X\| = \|\xi\|_X^2.$$

Definition 2.2. A representation (π, t) is said to be *injective* if a $*$-homomorphism π is injective.

By the above computation, we see that t is isometric for an injective representation (π, t).

Definition 2.3. For a representation (π, t) of a C^*-correspondence X on B, we define a $*$-homomorphism $\psi_t\colon \mathcal{K}(X) \to B$ by $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^* \in B$ for $\xi, \eta \in X$.

For the well-definedness of a $*$-homomorphism ψ_t, see, for example, [KPW, Lemma 2.2]. The following lemma is easily verified.

Lemma 2.4. *For a representation (π, t) of a C^*-correspondence X over A, we have $\pi(a)\psi_t(k) = \psi_t(\varphi_X(a)k)$ and $\psi_t(k)t(\xi) = t(k\xi)$ for $a \in A$, $\xi \in X$ and $k \in \mathcal{K}(X)$.*

By this lemma, we see that ψ_t is injective for an injective representation (π, t).

Definition 2.5. Let (π, t) be a representation of X. We set $t^0 = \pi$ and $t^1 = t$. For $n = 2, 3, \ldots$, we define a linear map $t^n\colon X^{\otimes n} \to C^*(\pi, t)$ by $t^n(\xi \otimes \eta) = t(\xi)t^{n-1}(\eta)$ for $\xi \in X$ and $\eta \in X^{\otimes(n-1)}$.

It is routine to see that t^n is well-defined and that (π, t^n) is a representation of the C^*-correspondence $X^{\otimes n}$. Hence we can define $\psi_{t^n}\colon \mathcal{K}(X^{\otimes n}) \to C^*(\pi, t)$ by $\psi_{t^n}(\theta_{\xi,\eta}) = t^n(\xi)t^n(\eta)^*$ for $\xi, \eta \in X^{\otimes n}$. Note that t^n and ψ_{t^n} are isometric if (π, t) is an injective representation.

Lemma 2.6. *Let (π, t) be a representation of X. Take $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$ for $n, m \in \mathbb{N}$ with $n \geq m$. Then we have $t^m(\eta)^* t^n(\xi) = t^{n-m}(\zeta)$ where $\zeta = \tau^m_{n-m}(\eta)^*\xi \in X^{\otimes(n-m)}$.*

Proof. When $m = 0$, this follows from the fact that (π, t^n) is a representation of the C^*-correspondence $X^{\otimes n}$. Let m be a positive integer. We may assume $\xi = \eta' \otimes \zeta'$ for $\eta' \in X^{\otimes m}$ and $\zeta' \in X^{\otimes(n-m)}$ because the linear span of such elements is dense in $X^{\otimes n}$. We have

$$\begin{aligned} t^m(\eta)^* t^n(\xi) &= t^m(\eta)^* t^m(\eta') t^{n-m}(\zeta') \\ &= \pi(\langle \eta, \eta' \rangle_{X^{\otimes m}}) t^{n-m}(\zeta') \\ &= t^{n-m}(\varphi_{n-m}(\langle \eta, \eta' \rangle_{X^{\otimes m}})\zeta'). \end{aligned}$$

On the other hand, we get

$$\tau^m_{n-m}(\eta)^*\xi = \tau^m_{n-m}(\eta)^*(\eta' \otimes \zeta') = \varphi_{n-m}(\langle \eta, \eta' \rangle_{X^{\otimes m}})\zeta'.$$

We are done. □

Proposition 2.7. *For a representation (π, t) of X, we have*

$$C^*(\pi, t) = \overline{\operatorname{span}}\{t^n(\xi)t^m(\eta)^* \mid \xi \in X^{\otimes n},\ \eta \in X^{\otimes m},\ n, m \in \mathbb{N}\}.$$

Proof. Clearly, the right hand side is a closed $*$-invariant linear space which contains the images of π and t, and is contained in $C^*(\pi, t)$. Hence all we have to do is to check that this set is closed under the multiplication, and this follows from Lemma 2.6. □

3. C^*-ALGEBRAS ASSOCIATED WITH C^*-CORRESPONDENCES

In this section, we give definitions of the C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ for a C^*-correspondence X.

Definition 3.1. Let X be a C^*-correspondence over a C^*-algebra A. We denote by $(\bar{\pi}_X, \bar{t}_X)$ the universal representation of X, and set $\mathcal{T}_X = C^*(\bar{\pi}_X, \bar{t}_X)$.

The universal representation $(\bar{\pi}_X, \bar{t}_X)$ can be obtained by taking a direct sum of sufficiently many representations. By the universality, for every representation (π, t) of X we have a surjection $\rho\colon \mathcal{T}_X \to C^*(\pi, t)$ with $\pi = \rho \circ \bar{\pi}_X$ and $t = \rho \circ \bar{t}_X$. This surjection will be called a natural surjection.

Definition 3.2. For a C^*-correspondence X over A , we define an ideal J_X of A by

$$\begin{aligned} J_X &= \varphi_X^{-1}\big(\mathcal{K}(X)\big) \cap \big(\ker \varphi_X\big)^{\perp} \\ &= \{a \in A \mid \varphi_X(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker \varphi_X\} \end{aligned}$$

Note that $J_X = \varphi_X^{-1}\big(\mathcal{K}(X)\big)$ when φ_X is injective. The ideal J_X is the largest ideal to which the restriction of φ_X is an injection into $\mathcal{K}(X)$. The ideal J_X has the following property.

Proposition 3.3. *Let X be a C^*-correspondence over a C^*-algebra A, and (π, t) be an injective representation of X. If $a \in A$ satisfies $\pi(a) \in \psi_t(\mathcal{K}(X))$, then we have $a \in J_X$ and $\pi(a) = \psi_t(\varphi_X(a))$.*

Proof. Take $a \in A$ with $\pi(a) \in \psi_t(\mathcal{K}(X))$. Let $k \in \mathcal{K}(X)$ be an element with $\pi(a) = \psi_t(k)$. For each $\xi \in X$, we have

$$t(\varphi_X(a)\xi) = \pi(a)t(\xi) = \psi_t(k)t(\xi) = t(k\xi).$$

Since t is injective, we have $\varphi_X(a)\xi = k\xi$ for every $\xi \in X$. This implies that $\varphi_X(a) = k \in \mathcal{K}(X)$. Thus we get $\pi(a) = \psi_t(\varphi_X(a))$. Take $b \in \ker \varphi_X$ and we will show that $ab = 0$. We get

$$\pi(ab) = \pi(a)\pi(b) = \psi_t(\varphi_X(a))\pi(b) = \psi_t(\varphi_X(a)\varphi_X(b)) = 0.$$

Since π is injective, we obtain $ab = 0$ as desired. Thus $a \in J_X$. □

The above proposition motivates the following definition.

Definition 3.4. A representation (π, t) is said to be *covariant* if we have $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J_X$.

Definition 3.5. For a C^*-correspondence X over a C^*-algebra A, the C^*-algebra $\mathcal{O}_X$ is defined by $\mathcal{O}_X = C^*(\pi_X, t_X)$ where (π_X, t_X) is the universal covariant representation of X.

By the universality, for each covariant representation (π, t) of a C^*-correspondence X, there exists a natural surjection $\rho\colon \mathcal{O}_X \to C^*(\pi, t)$ satisfying $\pi = \rho \circ \pi_X$ and $t = \rho \circ t_X$.

The construction of C^*-algebras $\mathcal{O}_X$ from C^*-correspondences X generalizes both the one in [P] for C^*-correspondences with injective left actions and the one in [AEE] for C^*-correspondences coming from Hilbert C^*-bimodules. This is also a generalization of the construction of graph algebras [KPRR, KPR, FLR] and more generally C^*-algebras arising from topological graphs [Ka1]. For the detail, see [Ka2].

4. The Fock representation

In this section, we construct a representation of a given C^*-correspondence, which is called the Fock representation. The Fock representation is injective, and from this we get an injective covariant representation. Most of the results in this section can be found in [P] or [MS]. We will need them in Sections 7 and 8. For the convenience of the readers, we give complete proofs.

Definition 4.1. The Hilbert A-module $\mathcal{F}(X)$, obtained as the direct sum of the Hilbert A-modules $X^{\otimes 0}, X^{\otimes 1}, \ldots$, is called the *Fock space* of X.

We consider $X^{\otimes n}$ as a submodule of $\mathcal{F}(X)$ for each $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$, we consider the space $\mathcal{L}(X^{\otimes n}, X^{\otimes m})$ of adjointable operators from $X^{\otimes n}$ to $X^{\otimes m}$ as a subspace of $\mathcal{L}(\mathcal{F}(X))$.

Definition 4.2. We define a $*$-homomorphism $\varphi_\infty\colon A \to \mathcal{L}(\mathcal{F}(X))$ and a linear map $\tau_\infty\colon X \to \mathcal{L}(\mathcal{F}(X))$ by

$$\varphi_\infty(a) = \sum_{m=0}^{\infty} \varphi_m(a), \qquad \tau_\infty(\xi) = \sum_{m=0}^{\infty} \tau_m^1(\xi),$$

for $a \in A$ and $\xi \in X$, where we always use the strong topology for the infinite sum of elements in $\mathcal{L}(\mathcal{F}(X))$.

Proposition 4.3 ([P, Proposition 1.3])**.** *The pair $(\varphi_\infty, \tau_\infty)$ is an injective representation of X on $\mathcal{L}(\mathcal{F}(X))$.*

Proof. By taking $n = 1$ in Lemma 1.9 (ii) and (iv), we see that $(\varphi_\infty, \tau_\infty)$ is a representation of X. It is injective because $\varphi_0\colon A \to \mathcal{L}(X^{\otimes 0})$ is an isomorphism onto $\mathcal{K}(X^{\otimes 0})$. $\square$

This representation $(\varphi_\infty, \tau_\infty)$ is called the *Fock representation*. From the Fock representation $(\varphi_\infty, \tau_\infty)$, we can define a linear map $\tau_\infty^n\colon X^{\otimes n} \to \mathcal{L}(\mathcal{F}(X))$ for each $n \in \mathbb{N}$ as in Definition 2.5. It is easy to see that $\tau_\infty^n(\xi) = \sum_{m=0}^{\infty} \tau_m^n(\xi)$ for $\xi \in X^{\otimes n}$ and $n \in \mathbb{N}$.

Proposition 4.4. *For $a \in J_X$, we have*

$$\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_X(a)) = \varphi_0(a) \in \mathcal{L}(X^{\otimes 0}) \subset \mathcal{L}(\mathcal{F}(X)).$$

Proof. For $\xi, \eta \in X$, we have $\psi_{\tau_\infty}(\theta_{\xi,\eta}) = \sum_{m=1}^{\infty} \theta_{\xi,\eta} \otimes \mathrm{id}_{m-1}$ by Lemma 1.9 (i). Hence we have $\psi_{\tau_\infty}(k) = \sum_{m=1}^{\infty} k \otimes \mathrm{id}_{m-1}$ for all $k \in \mathcal{K}(X)$. Therefore we obtain

$$\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_X(a)) = \sum_{m=0}^{\infty} \varphi_m(a) - \sum_{m=1}^{\infty} \varphi_X(a) \otimes \mathrm{id}_{m-1} = \varphi_0(a)$$

because $\varphi_m(a) = \varphi_X(a) \otimes \mathrm{id}_{m-1}$ for $m \geq 1$. $\square$

Corollary 4.5. *If $a \in A$ satisfies $\varphi_\infty(a) \in \psi_{\tau_\infty}(\mathcal{K}(X))$, then $a = 0$.*

Proof. For $a \in A$ with $\varphi_\infty(a) \in \psi_{\tau_\infty}(\mathcal{K}(X))$, we have $a \in J_X$ and $\varphi_\infty(a) = \psi_{\tau_\infty}(\varphi_X(a))$ by Proposition 3.3. By Proposition 4.4, we get $\varphi_0(a) = \varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_X(a)) = 0$. Thus we obtain $a = 0$ because φ_0 is injective. $\square$

The set $\mathcal{F}(X)J_X$ is a Hilbert J_X-module ([Ka3, Corollary 1.4]), and we have

$$\mathcal{K}(\mathcal{F}(X)J_X) = \overline{\operatorname{span}}\{\theta_{\xi a,\eta} \in \mathcal{K}(\mathcal{F}(X)) \mid \xi, \eta \in \mathcal{F}(X), a \in J_X\},$$

which is an ideal of $\mathcal{L}(\mathcal{F}(X))$. We see that $k \in \mathcal{K}(\mathcal{F}(X))$ is in $\mathcal{K}(\mathcal{F}(X)J_X)$ if and only if $\langle \xi, k\eta \rangle \in J_X$ for all $\xi, \eta \in \mathcal{F}(X)$ (see [FMR, Lemma 2.6] or [Ka3, Lemma 1.6]).

Proposition 4.6. *We have $\mathcal{K}(\mathcal{F}(X)J_X) \subset C^*(\varphi_\infty, \tau_\infty)$.*

Proof. For $\xi \in X^{\otimes n}, \eta \in X^{\otimes m}$ and $a \in J_X$, we have

$$\theta_{\xi a,\eta} = \tau_\infty^n(\xi)\varphi_0(a)\tau_\infty^m(\eta)^* = \tau_\infty^n(\xi)\big(\varphi_\infty(a) - \psi_{\tau_\infty}(\varphi_X(a))\big)\tau_\infty^m(\eta)^* \in C^*(\varphi_\infty, \tau_\infty)$$

by Proposition 4.4. Hence $\mathcal{K}(\mathcal{F}(X)J_X) \subset C^*(\varphi_\infty, \tau_\infty)$. $\square$

Let $\sigma\colon \mathcal{L}(\mathcal{F}(X)) \to \mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X)$ be the quotient map, and set $\varphi = \sigma \circ \varphi_\infty$ and $\tau = \sigma \circ \tau_\infty$. By Proposition 4.4, (φ, τ) is a covariant representation of X on $\mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X)$. We will see that this representation (φ, τ) is injective.

Lemma 4.7. *For $n \geq 1$, the restriction of the $*$-homomorphism $\mathcal{L}(X^{\otimes n}) \ni S \mapsto S \otimes \mathrm{id}_1 \in \mathcal{L}(X^{\otimes(n+1)})$ to $\mathcal{K}(X^{\otimes n}J_X)$ is injective.*

Proof. Take $k \in \mathcal{K}(X^{\otimes n}J_X)$ with $k \otimes \mathrm{id}_1 = 0$. Then for all $\xi, \xi' \in X^{\otimes n}$ and all $\eta, \eta' \in X$, we have

$$0 = \langle \xi \otimes \eta, (k \otimes \mathrm{id}_1)(\xi' \otimes \eta') \rangle_{X^{\otimes(n+1)}} = \big\langle \eta, \varphi_X(\langle \xi, k\xi' \rangle_{X^{\otimes n}})\eta' \big\rangle_X.$$

Hence we have $\varphi_X(\langle \xi, k\xi' \rangle_{X^{\otimes n}}) = 0$ for all $\xi, \xi' \in X^{\otimes n}$. Since $k \in \mathcal{K}(X^{\otimes n}J_X)$, we have $\langle \xi, k\xi' \rangle_{X^{\otimes n}} \in J_X$. Thus $\langle \xi, k\xi' \rangle_{X^{\otimes n}} = 0$ for all $\xi, \xi' \in X^{\otimes n}$ because φ_X is injective on J_X. Therefore we get $k = 0$. Thus the restriction of the map $S \mapsto S \otimes \mathrm{id}_1$ to $\mathcal{K}(X^{\otimes n}J_X)$ is injective. □

Lemma 4.8. *For $a \in A$, $\varphi_\infty(a) \in \mathcal{K}(\mathcal{F}(X))$ implies $\lim_{n\to\infty} \|\varphi_n(a)\| = 0$.*

Proof. For each $n \in \mathbb{N}$, let $P_n \in \mathcal{L}(\mathcal{F}(X))$ be the projection onto the direct summand $X^{\otimes n} \subset \mathcal{F}(X)$. Since $\varphi_n(a) = P_n\varphi_\infty(a)P_n$, it suffices to show that $\lim_{n\to\infty} \|P_n k P_n\| = 0$ for each $k \in \mathcal{K}(\mathcal{F}(X))$. We may assume $k = \theta_{\xi,\eta}$ for $\xi, \eta \in \mathcal{F}(X)$ because the linear span of such elements is dense in $\mathcal{K}(\mathcal{F}(X))$. By the same reason, we may assume $\xi \in X^{\otimes k}$ and $\eta \in X^{\otimes l}$ for some $k, l \in \mathbb{N}$. Now it is clear that we have $\lim_{n\to\infty} \|P_n k P_n\| = 0$. This completes the proof. □

Proposition 4.9. *The covariant representation (φ, τ) is injective.*

Proof. Take $a \in A$ with $\varphi(a) = 0$. Then we have $\varphi_\infty(a) \in \mathcal{K}(\mathcal{F}(X)J_X)$. For each $n \in \mathbb{N}$, we have

$$\varphi_n(a) = P_n\varphi_\infty(a)P_n \in P_n\mathcal{K}(\mathcal{F}(X)J_X)P_n = \mathcal{K}(X^{\otimes n}J_X)$$

where $P_n \in \mathcal{L}(\mathcal{F}(X))$ is the projection onto the direct summand $X^{\otimes n} \subset \mathcal{F}(X)$. By taking $n = 0$, we get $a \in J_X$. Since $\varphi_1 = \varphi_X$ is injective on J_X, we have $\|a\| = \|\varphi_1(a)\|$. By Lemma 4.7, we have $\|\varphi_n(a)\| = \|\varphi_n(a) \otimes \mathrm{id}_1\| = \|\varphi_{n+1}(a)\|$ for all positive integer n. Therefore we get $\|\varphi_n(a)\| = \|a\|$ for all $n \in \mathbb{N}$. Thus we have $a = 0$ by Lemma 4.8. This proves that the covariant representation (φ, τ) is injective. □

As consequences of Corollary 4.5 and Proposition 4.9, we have the followings.

Proposition 4.10. *The universal representation $(\bar{\pi}_X, \bar{t}_X)$ of X on $\mathcal{T}_X$ satisfies that $\{a \in A \mid \bar{\pi}_X(a) \in \psi_{\bar{t}_X}(\mathcal{K}(X))\} = 0$.*

Proposition 4.11. *The universal covariant representation (π_X, t_X) of X on $\mathcal{O}_X$ is injective.*

We will see in Section 6 that the Fock representation $(\varphi_\infty, \tau_\infty)$ is the universal representation, and (φ, τ) is the universal covariant representation.

Note that the C^*-algebra $C^*(\varphi_\infty, \tau_\infty)$ is the augmented Cuntz-Toeplitz algebra defined in [P], and the C^*-algebra $C^*(\varphi, \tau)$ is the relative Cuntz-Pimsner algebra $\mathcal{O}(J_X, X)$ defined in [MS, Definition 2.18].

5. Analysis of the cores

In this section, we investigate the so-called cores of C^*-algebras $C^*(\pi, t)$ for representations (π, t) of a C^*-correspondence X. Fix a C^*-correspondence X over a C^*-algebra A, and a representation (π, t) of X.

Definition 5.1. For each $n \in \mathbb{N}$, we set $B_n = \psi_{t^n}\big(\mathcal{K}(X^{\otimes n})\big) \subset C^*(\pi, t)$.

Note that $B_0 = \pi(A)$ and that $B_n \cong \mathcal{K}(X^{\otimes n})$ when (π, t) is injective. We can easily see the next lemma.

Lemma 5.2. *For $n, m \in \mathbb{N}$ with $n \geq 1$, we have $\overline{\operatorname{span}}(t^n(X^{\otimes n})B_m t^n(X^{\otimes n})^*) = B_{n+m}$ and $t^n(X^{\otimes n})^* B_{n+m} t^n(X^{\otimes n}) \subset B_m$.*

Definition 5.3. For $m, n \in \mathbb{N}$ with $m \leq n$, we define $B_{[m,n]} \subset C^*(\pi, t)$ by $B_{[m,n]} = B_m + B_{m+1} + \cdots + B_n$.

We have $B_{[n,n]} = B_n$ for each $n \in \mathbb{N}$. By the next lemma, we see that $B_{[m,n]}$'s are C^*-subalgebras of $C^*(\pi, t)$.

Lemma 5.4. *For $m, n \in \mathbb{N}$ with $m \leq n$, $k \in \mathcal{K}(X^{\otimes m})$ and $k' \in \mathcal{K}(X^{\otimes n})$, we have $\psi_{t^m}(k)\psi_{t^n}(k') = \psi_{t^n}((k \otimes \mathrm{id}_{n-m})k')$.*

Proof. It suffices to show that $\psi_{t^m}(k)t^n(\xi) = t^n((k \otimes \mathrm{id}_{n-m})\xi)$ for $k \in \mathcal{K}(X^{\otimes m})$ and $\xi \in X^{\otimes n}$. When $m = 0$, this equation follows from the fact that (π, t^n) is a representation of the C^*-correspondence $X^{\otimes n}$. Suppose $m \geq 1$. We may assume $k = \theta_{\zeta,\eta}$ for $\zeta, \eta \in X^{\otimes m}$. We have

$$\begin{aligned}
\psi_{t^m}(k)t^n(\xi) &= t^m(\zeta)t^m(\eta)^*t^n(\xi) \\
&= t^m(\zeta)t^{n-m}\big(\tau^m_{n-m}(\eta)^*\xi\big) \\
&= t^n\big(\zeta \otimes (\tau^m_{n-m}(\eta)^*\xi)\big) \\
&= t^n\big((\tau^m_{n-m}(\zeta)\tau^m_{n-m}(\eta)^*)\xi\big) \\
&= t^n((k \otimes \mathrm{id}_{n-m})\xi)
\end{aligned}$$

by Lemma 2.6 and Lemma 1.9 (i). We are done. $\square$

By the above lemma, $B_{[k,n]}$ is an ideal of $B_{[m,n]}$ for $m, k, n \in \mathbb{N}$ with $m \leq k \leq n$. In particular, B_n is an ideal of $B_{[0,n]}$ for each $n \in \mathbb{N}$.

Definition 5.5. For $m \in \mathbb{N}$, we define a C^*-subalgebra $B_{[m,\infty]}$ of $C^*(\pi, t)$ by $B_{[m,\infty]} = \overline{\bigcup_{n=m}^{\infty} B_{[m,n]}}$.

Note that the C^*-algebra $B_{[m,\infty]}$ is an inductive limit of the increasing sequence of C^*-algebras $\{B_{[m,n]}\}_{n=m}^{\infty}$. The C^*-algebra $B_{[0,\infty]}$ is called the *core* of the C^*-algebra $C^*(\pi, t)$. The core $B_{[0,\infty]}$ naturally arises when the C^*-algebra $C^*(\pi, t)$ has an action of $\mathbb{T}$ called a gauge action.

Definition 5.6. A representation (π, t) of X is said to *admit a gauge action* if for each $z \in \mathbb{T}$, there exists a $*$-homomorphism $\beta_z \colon C^*(\pi, t) \to C^*(\pi, t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = zt(\xi)$ for all $a \in A$ and $\xi \in X$.

If it exists, such a $*$-homomorphism β_z is unique. By the assumptions in the definition above, β_z is a $*$-automorphism for all $z \in \mathbb{T}$ and the map $\beta \colon \mathbb{T} \to \mathrm{Aut}(C^*(\pi, t))$ is automatically a strongly continuous homomorphism. By the universality, both the universal representation $(\bar{\pi}_X, \bar{t}_X)$ on $\mathcal{T}_X$ and the universal covariant representation (π_X, t_X) on $\mathcal{O}_X$ admit gauge actions. We denote these actions by $\bar{\gamma} \colon \mathbb{T} \curvearrowright \mathcal{T}_X$ and $\gamma \colon \mathbb{T} \curvearrowright \mathcal{O}_X$. It is clear that for a representation (π, t) admitting a gauge action β we have $\beta_z \circ \rho = \rho \circ \bar{\gamma}_z$ for each $z \in \mathbb{T}$, where $\rho \colon \mathcal{T}_X \to C^*(\pi, t)$ is the natural surjection.

It is also clear that for a covariant representation (π, t) admitting a gauge action β we have $\beta_z \circ \rho = \rho \circ \gamma_z$ for each $z \in \mathbb{T}$, where $\rho\colon \mathcal{O}_X \to C^*(\pi, t)$ is the natural surjection.

Proposition 5.7. *When a representation (π, t) admits a gauge action β, the core $B_{[0,\infty]}$ coincides with the fixed point algebra $C^*(\pi, t)^\beta$.*

Proof. Since

$$\beta_z\big(t^n(\xi)t^m(\eta)^*\big) = z^{n-m}t^n(\xi)t^m(\eta)^*$$

for $\xi \in X^{\otimes n}$, $\eta \in X^{\otimes m}$ and $z \in \mathbb{T}$, it is clear that $B_{[0,\infty]} \subset C^*(\pi, t)^\beta$. Take $x \in C^*(\pi, t)^\beta$. By Proposition 2.7, there exists a sequence $\{x_k\}_{k=0}^\infty$ of linear sums of elements in the form $t^n(\xi)t^m(\eta)^*$ such that $x = \lim_{k\to\infty} x_k$. Then we have

$$x = \int_{\mathbb{T}} \beta_z(x)dz = \lim_{k\to\infty} \int_{\mathbb{T}} \beta_z(x_k)dz$$

where dz is the normalized Haar measure on $\mathbb{T}$. By the above computation, we get $\int_{\mathbb{T}} \beta_z(x_k)dz \in \bigcup_{n=0}^\infty B_n$ for every k. Thus we have $x \in B_{[0,\infty]}$. We have shown that $B_{[0,\infty]} = C^*(\pi, t)^\beta$. □

We are going to compute the core $B_{[0,\infty]} \subset C^*(\pi, t)$. To do this end, we need the following notation.

Definition 5.8. For a representation (π, t) of X, we set

$$I'_{(\pi,t)} = \{a \in A \mid \pi(a) \in B_1 = \psi_t(\mathcal{K}(X))\},$$

which is an ideal of A. For each $n \in \mathbb{N}$, we define

$$B'_n = \psi_{t^n}\big(\mathcal{K}(X^{\otimes n}I'_{(\pi,t)})\big) \subset B_n \subset C^*(\pi, t).$$

Proposition 5.9. *For each $n \in \mathbb{N}$, we have $B_n \cap B_{n+1} = B'_n$.*

Proof. The case $n = 0$ follows from the definition of $I'_{(\pi,t)}$. Let n be a positive integer. For $a \in I'_{(\pi,t)}$ and $\xi, \eta \in X^{\otimes n}$, we have

$$\psi_{t^n}(\theta_{\xi a,\eta}) = t^n(\xi a)t^n(\eta)^* = t^n(\xi)\pi(a)t^n(\eta)^* \in B_{n+1}$$

because $\pi(a) \in B_1$. Hence we get $B'_n \subset B_n \cap B_{n+1}$. Conversely take $x \in B_n \cap B_{n+1}$. Take $k \in \mathcal{K}(X^{\otimes n})$ with $\psi_{t^n}(k) = x$. For each $\xi, \eta \in X^{\otimes n}$, we have

$$\pi(\langle \xi, k\eta\rangle_X) = t^n(\xi)^*\psi_{t^n}(k)t^n(\eta) = t^n(\xi)^*xt^n(\eta) \in B_1$$

because $x \in B_{n+1}$. This implies that $\langle \xi, k\eta\rangle_X \in I'_{(\pi,t)}$ for all $\xi, \eta \in X^{\otimes n}$. Hence we have $k \in \mathcal{K}(X^{\otimes n}I'_{(\pi,t)})$. Thus we get $x = \psi_{t^n}(k) \in B'_n$. We have shown $B_n \cap B_{n+1} = B'_n$ for all $n \in \mathbb{N}$. □

Lemma 5.10. *Let n be a positive integer. For an approximate unit $\{u_\lambda\}$ of $\mathcal{K}(X^{\otimes n})$ and $k \in \mathcal{K}(X^{\otimes(n+1)})$, we have $k = \lim_\lambda (u_\lambda \otimes \mathrm{id}_1)k$.*

Proof. Clearly the equality holds for $k = (k' \otimes \mathrm{id}_1)k'' \in \mathcal{K}(X^{\otimes(n+1)})$ where $k' \in \mathcal{K}(X^{\otimes n})$ and $k'' \in \mathcal{K}(X^{\otimes(n+1)})$. We will show that the linear span of such elements is dense in $\mathcal{K}(X^{\otimes(n+1)})$. To do so, it suffices to show that the linear span of elements in the form $(k' \otimes \mathrm{id}_1)\zeta$ with $k' \in \mathcal{K}(X^{\otimes n})$ and $\zeta \in X^{\otimes(n+1)}$ is dense in $X^{\otimes(n+1)}$

because we have $(k'\otimes \mathrm{id}_1)\theta_{\zeta,\zeta'} = \theta_{(k'\otimes\mathrm{id}_1)\zeta,\zeta'}$. For $k'=\theta_{\xi,\xi'}$ and $\zeta=\eta\otimes\eta'$ with $\xi,\xi',\eta\in X^{\otimes n}$ and $\eta'\in X$, we have

$$\begin{aligned}(k'\otimes \mathrm{id}_1)\zeta &= \tau_1^n(\xi)\tau_1^n(\xi')^*(\eta\otimes\eta')\\ &= \tau_1^n(\xi)(\varphi_1(\langle\xi',\eta\rangle_{X^{\otimes n}})\eta')\\ &= \xi\otimes(\varphi_X(\langle\xi',\eta\rangle_{X^{\otimes n}})\eta')\\ &= \xi\langle\xi',\eta\rangle_{X^{\otimes n}}\otimes\eta'.\end{aligned}$$

Since the linear span of elements in the form $\xi\langle\xi',\eta\rangle_{X^{\otimes n}}$ with $\xi,\xi',\eta\in X^{\otimes n}$ is dense in $X^{\otimes n}$ and the linear span of elements in the form $\xi\otimes\eta'$ with $\xi\in X^{\otimes n}$ and $\eta'\in X$, is dense in $X^{\otimes(n+1)}$, we see that the linear span of elements in the form $(k'\otimes\mathrm{id}_1)\zeta$ with $k'\in\mathcal{K}(X^{\otimes n})$ and $\zeta\in X^{\otimes(n+1)}$ is dense in $X^{\otimes(n+1)}$. We are done. $\square$

Proposition 5.11. *For each $n\in\mathbb{N}$, we have $B_{[0,n]}\cap B_{n+1}\subset B_n$.*

Proof. The assertion is obvious for $n=0$. We assume $n\geq 1$. Take $x\in B_{[0,n]}\cap B_{n+1}$. Choose $k\in\mathcal{K}(X^{\otimes(n+1)})$ such that $x=\psi_{t^{n+1}}(k)$. For an approximate unit $\{u_\lambda\}$ of $\mathcal{K}(X^{\otimes n})$, we have $k=\lim_\lambda(u_\lambda\otimes\mathrm{id}_1)k$ by Lemma 5.10. Since $\psi_{t^n}(u_\lambda)\psi_{t^{n+1}}(k) = \psi_{t^{n+1}}\big((u_\lambda\otimes\mathrm{id}_1)k\big)$ by Lemma 5.4, we see

$$x=\psi_{t^{n+1}}(k)=\lim_\lambda\psi_{t^n}(u_\lambda)\psi_{t^{n+1}}(k)=\lim_\lambda\psi_{t^n}(u_\lambda)x.$$

Since B_n is an ideal of $B_{[0,n]}$, we have $x\in B_n$. Thus we obtain $B_{[0,n]}\cap B_{n+1}\subset B_n$. $\square$

Proposition 5.12. *For each $n\in\mathbb{N}$, we have $B_{[0,n]}\cap B_{n+1}=B'_n$, and we get the following commutative diagram with exact rows;*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B'_n & \longrightarrow & B_{[0,n]} & \longrightarrow & B_{[0,n]}/B'_n & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_{n+1} & \longrightarrow & B_{[0,n+1]} & \longrightarrow & B_{[0,n]}/B'_n & \longrightarrow & 0.\end{array}$$

Proof. The former part follows from Proposition 5.9 and Proposition 5.11. The latter part follows from the former and the fact $B_{[0,n+1]}=B_{[0,n]}+B_{n+1}$. $\square$

Proposition 5.13. *For $n=1,2,\ldots,\infty$, we have the following short exact sequences;*

$$0\longrightarrow B_{[1,n]}\longrightarrow B_{[0,n]}\longrightarrow B_0/B'_0\longrightarrow 0.$$

Proof. We will first prove $B_0\cap B_{[1,n]}=B'_0$ by the induction with respect to n. The case that $n=1$ follows from Proposition 5.9. Suppose that we have proved $B_0\cap B_{[1,n]}=B'_0$. Take $x\in B_0\cap B_{[1,n+1]}$. Choose $y\in B_{[1,n]}$ and $z\in B_{n+1}$ with $x=y+z$. We have $z=x-y\in B_{[0,n]}\cap B_{n+1}$. By Proposition 5.11, we have $z\in B_n$. Thus $x=y+z\in B_{[1,n]}$. Hence we have shown $B_0\cap B_{[1,n+1]}\subset B_0\cap B_{[1,n]}$. Since the converse inclusion is obvious, we get $B_0\cap B_{[1,n+1]}=B_0\cap B_{[1,n]}=B'_0$. Thus we obtain $B_0\cap B_{[1,n]}=B'_0$ for all positive integer n. This implies the existence of the desired short exact sequences for $n=1,2,\ldots$, because $B_{[0,n]}=B_{[1,n]}+B_0$. By taking inductive limits, we obtain the short exact sequences for $n=\infty$. $\square$

The C^*-subalgebras of $\mathcal{T}_X$ and $\mathcal{O}_X$ corresponding to $B_n, B_{[m,n]}$ are denoted by $\bar{\mathcal{B}}_n,\bar{\mathcal{B}}_{[m,n]}\subset\mathcal{T}_X$ and $\mathcal{B}_n,\mathcal{B}_{[m,n]}\subset\mathcal{O}_X$. By Proposition 5.7 we have $\mathcal{T}_X^{\bar\gamma}=\bar{\mathcal{B}}_{[0,\infty]}$ and $\mathcal{O}_X^\gamma=\mathcal{B}_{[0,\infty]}$.

Proposition 5.14. *There exists a short exact sequence*

$$0 \longrightarrow \bar{\mathcal{B}}_{n+1} \longrightarrow \bar{\mathcal{B}}_{[0,n+1]} \longrightarrow \bar{\mathcal{B}}_{[0,n]} \longrightarrow 0,$$

which splits by the natural inclusion $\bar{\mathcal{B}}_{[0,n]} \hookrightarrow \bar{\mathcal{B}}_{[0,n+1]}$.

Proof. This follows from Proposition 5.12 because Proposition 4.10 implies $I'_{(\bar{\pi}_X,\bar{t}_X)} = 0$. □

Proposition 5.15. *There exists a surjection from* $\mathcal{T}_X^{\bar{\gamma}}$ *to* A.

Proof. This follows from Proposition 5.13. □

Proposition 5.16. *We get the following commutative diagram with exact rows;*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{B}_{[1,n+1]} & \longrightarrow & \mathcal{B}_{[0,n+1]} & \longrightarrow & A/J_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{B}_{[1,\infty]} & \longrightarrow & \mathcal{B}_{[0,\infty]} & \longrightarrow & A/J_X & \longrightarrow & 0.
\end{array}$$

Proof. By noting that $\mathcal{B}_0 \cong A$ and $\mathcal{B}'_0 \cong J_X$, this follows from Proposition 5.13. □

Proposition 5.17. *We get the following commutative diagram with exact rows;*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_X & \longrightarrow & A & \longrightarrow & A/J_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_X} & & \| & & \\
0 & \longrightarrow & \mathcal{B}_{[1,\infty]} & \longrightarrow & \mathcal{O}_X^{\gamma} & \longrightarrow & A/J_X & \longrightarrow & 0.
\end{array}$$

Proof. This follows from Proposition 5.13. □

Proposition 5.18. *For a C^*-correspondence X over a C^*-algebra A, the following conditions are equivalent;*

(i) *the injection* $\pi_X\colon A \to \mathcal{O}_X^{\beta}$ *is an isomorphism,*
(ii) *we have* $\mathcal{B}_0 \supset \mathcal{B}_1$,
(iii) *the injection* $\varphi_X\colon J_X \to \mathcal{K}(X)$ *is an isomorphism,*
(iv) *the C^*-correspondence X comes from a Hilbert A-bimodule.*

Proof. It is clear that (i) implies (ii). From the condition (ii), we obtain $\mathcal{B}_n \supset \mathcal{B}_{n+1}$ for all $n \in \mathbb{N}$ by Lemma 5.2. Hence (ii) implies $\mathcal{O}_X^{\beta} = \mathcal{B}_0 = \pi_X(A)$. This shows the implication (ii) ⇒ (i). By setting $n = 0$ in Proposition 5.12, we have the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_X & \xrightarrow[\pi_X]{} & \mathcal{B}_0 & \longrightarrow & A/J_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_X} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}(X) & \xrightarrow[\psi_{t_X}]{} & \mathcal{B}_{[0,1]} & \longrightarrow & A/J_X & \longrightarrow & 0.
\end{array}$$

From this diagram, we have the equivalence (ii) $\Longleftrightarrow$ (iii). Finally, the equivalence (iii) $\Longleftrightarrow$ (iv) was shown in [Ka2]. □

6. The gauge-invariant uniqueness theorems

In this section, we will give conditions for representations or covariant representations to be universal. The idea of the proof can be seen in [Ka1, Section 4] (and also in [P, Section 3], [FMR, Section 4]). Let us take a C^*-correspondence X over a C^*-algebra A.

Proposition 6.1. *For a representation (π, t) of X satisfying $I'_{(\pi,t)} = 0$, the restriction of $\rho\colon \mathcal{T}_X \to C^*(\pi, t)$ to the fixed point algebra $\mathcal{T}_X^{\bar{\gamma}}$ is injective.*

Proof. For $n \in \mathbb{N}$ let B_n and $B_{[0,n]}$ be C^*-subalgebras of $C^*(\pi, t)$ defined in Definition 5.1 and Definition 5.3. From the condition $I'_{(\pi,t)} = 0$, we get the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{\mathcal{B}}_{n+1} & \longrightarrow & \bar{\mathcal{B}}_{[0,n+1]} & \longrightarrow & \bar{\mathcal{B}}_{[0,n]} & \longrightarrow & 0 \\
 & & \big\downarrow \rho & & \big\downarrow \rho & & \big\downarrow \rho & & \\
0 & \longrightarrow & B_{n+1} & \longrightarrow & B_{[0,n+1]} & \longrightarrow & B_{[0,n]} & \longrightarrow & 0
\end{array}$$

by the same argument as in Proposition 5.14. Since the condition $I'_{(\pi,t)} = 0$ implies that the representation (π, t) is injective, we see that the restriction of ρ to $\bar{\mathcal{B}}_n$ is injective for all $n \in \mathbb{N}$. By using this fact and the commutative diagram above, we can inductively show that the restriction of ρ to $\bar{\mathcal{B}}_{[0,n]}$ is injective. Hence the restriction of ρ to $\mathcal{T}_X^{\bar{\gamma}} = \bar{\mathcal{B}}_{[0,\infty]}$ is injective. □

The following is the gauge-invariant uniqueness theorem for the C^*-algebra $\mathcal{T}_X$.

Theorem 6.2. *Let X be a C^*-correspondence over a C^*-algebra A. For a representation (π, t) of X, the surjection $\rho\colon \mathcal{T}_X \to C^*(\pi, t)$ is an isomorphism if and only if (π, t) satisfies $I'_{(\pi,t)} = 0$ and admits a gauge action.*

Proof. We had already seen that the two conditions are necessary. Now suppose that a representation (π, t) admits a gauge action β, and satisfies $I'_{(\pi,t)} = 0$. Take $x \in \mathcal{T}_X$ with $\rho(x) = 0$. Then we have

$$\rho\Big(\int_{\mathbb{T}} \bar{\gamma}_z(x^*x)dz\Big) = \int_{\mathbb{T}} \rho(\bar{\gamma}_z(x^*x))dz = \int_{\mathbb{T}} \beta_z(\rho(x^*x))dz = 0,$$

where dz is the normalized Haar measure on $\mathbb{T}$. Since $\int_{\mathbb{T}} \bar{\gamma}_z(x^*x)dz \in \mathcal{T}_X^{\gamma}$, we have $\int_{\mathbb{T}} \bar{\gamma}_z(x^*x)dz = 0$ by Proposition 6.1. This implies $x^*x = 0$. Hence ρ is injective. □

Proposition 6.3. *For an injective covariant representation (π, t) of X, the restriction of the surjection $\rho\colon \mathcal{O}_X \to C^*(\pi, t)$ to the fixed point algebra $\mathcal{O}_X^{\gamma}$ is injective.*

Proof. For $n \in \mathbb{N}$ let B_n and $B_{[0,n]}$ be C^*-subalgebras of $C^*(\pi, t)$ defined in Definition 5.1 and Definition 5.3. Since ψ_{t^n} is injective, the restriction of ρ to $\mathcal{B}_n$ is an isomorphism onto B_n. It is easy to see that the restriction of ρ to $\mathcal{B}_{[0,n]}$ is a surjection onto $B_{[0,n]}$ for each $n \in \mathbb{N}$. We will show that these are injective by the induction with respect to n. The case that $n = 0$ follows from the fact that π is injective. Suppose that we had shown that the restriction of ρ to $\mathcal{B}_{[0,n]}$ is an isomorphism onto $B_{[0,n]}$. By Proposition 3.3, we have $I'_{(\pi_X,t_X)} = I'_{(\pi,t)} = J_X$. Hence the restriction of ρ to $\mathcal{B}'_n$

is an isomorphism onto B'_n. Thus we get an isomorphism $\mathcal{B}_{[0,n]}/\mathcal{B}'_n \to B_{[0,n]}/B'_n$. By Proposition 5.12 we get the following commutative diagram with exact rows;

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B}_{n+1} \arrow[r] \arrow[d, "\rho"] & \mathcal{B}_{[0,n+1]} \arrow[r] \arrow[d, "\rho"] & \mathcal{B}_{[0,n]}/\mathcal{B}'_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{n+1} \arrow[r] & B_{[0,n+1]} \arrow[r] & B_{[0,n]}/B'_n \arrow[r] & 0.
\end{tikzcd}$$

By the 5-lemma, we see that the surjection $\mathcal{B}_{[0,n+1]} \to B_{[0,n+1]}$ is an isomorphism. Thus we have shown that the restriction of ρ to $\mathcal{B}_{[0,n]}$ is injective for all $n \in \mathbb{N}$. Hence the restriction of ρ to $\mathcal{O}_X^\gamma = \mathcal{B}_{[0,\infty]}$ is injective. □

The following is the gauge-invariant uniqueness theorem for the C^*-algebra $\mathcal{O}_X$.

Theorem 6.4. *For a covariant representation (π, t) of a C^*-correspondence X, the $*$-homomorphism $\rho\colon \mathcal{O}_X \to C^*(\pi, t)$ is an isomorphism if and only if (π, t) is injective and admits a gauge action.*

Proof. The proof goes similarly as in Theorem 6.2 with the help of Proposition 6.3. □

When the left actions of C^*-correspondences are injective, Theorem 6.4 is the gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras which was proved in [FMR, Theorem 4.1]. In the case that C^*-correspondences are defined from graphs with or without sinks, this was already proved in [BHRS, Theorem 2.1]. For C^*-algebras arising from topological graphs, this was proved in [Ka1, Theorem 4.5].

We can apply the two gauge-invariant uniqueness theorems to the representations $(\varphi_\infty, \tau_\infty)$ and (φ, τ) in Section 4.

Proposition 6.5. *Both the representation $(\varphi_\infty, \tau_\infty)$ and the covariant representation (φ, τ) are universal, that is, we have natural isomorphisms $C^*(\varphi_\infty, \tau_\infty) \cong \mathcal{T}_X$ and $C^*(\varphi, \tau) \cong \mathcal{O}_X$.*

Proof. To apply Theorem 6.2 and Theorem 6.4, it suffices to see that both of the representations $(\varphi_\infty, \tau_\infty)$ and (φ, τ) admit gauge actions because the other conditions had already been checked in Section 4.

For each $z \in \mathbb{T}$, define a unitary $u_z \in \mathcal{L}(\mathcal{F}(X))$ by $u_z(\xi) = z^n\xi$ for $\xi \in X^{\otimes n} \subset \mathcal{F}(X)$ and $n \in \mathbb{N}$. It is routine to see that the automorphisms $\operatorname{Ad} u_z$ of $\mathcal{L}(\mathcal{F}(X))$, defined by $\operatorname{Ad} u_z(x) = u_z x u_z^*$ for $x \in \mathcal{L}(\mathcal{F}(X))$, give a gauge action for the representation $(\varphi_\infty, \tau_\infty)$. The ideal $\mathcal{K}(\mathcal{F}(X)J_X)$ of $\mathcal{L}(\mathcal{F}(X))$ is closed under the automorphisms $\operatorname{Ad} u_z$ for each $z \in \mathbb{T}$. Hence we can define an automorphism β_z of $\mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X)$ by $\beta_z(\sigma(x)) = \sigma(u_z x u_z^*)$ for $x \in \mathcal{L}(\mathcal{F}(X))$ and $z \in \mathbb{T}$. It is clear that β is a gauge action for the representation (φ, τ). We are done. □

By Proposition 6.5, the C^*-algebra $\mathcal{O}_X$ is isomorphic to the relative Cuntz-Pimsner algebras $C^*(\varphi, \tau) = \mathcal{O}(J_X, X)$ introduced in [MS] (cf. [MS, Theorem 2.19]). The isomorphism $C^*(\varphi_\infty, \tau_\infty) \cong \mathcal{T}_X$ was already proved in [P, Theorem 3.4] under small assumption on C^*-correspondences.

The C^*-algebra $\mathcal{O}_X$ was defined as the largest C^*-algebra among C^*-algebras $C^*(\pi, t)$ generated by covariant representations (π, t) of X. Theorem 6.4 tells us that we have $C^*(\pi, t) \cong \mathcal{O}_X$ when a covariant representation (π, t) satisfies two conditions; being injective and admitting a gauge action. In the next paper [Ka3],

we will see that the C^*-algebra $\mathcal{O}_X$ can be defined as the smallest C^*-algebra among C^*-algebras $C^*(\pi,t)$ generated by representations (π,t) of X which satisfy the two conditions above; being injective and admitting gauge actions. Thus we can define $\mathcal{O}_X$ without using the ideal J_X.

7. Nuclearity and exactness

In this section, we study when the C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ become nuclear or exact. We use the facts on nuclearity and exactness appeared in Appendices A and B as well as in [W].

On the exactness of $\mathcal{T}_X$ and $\mathcal{O}_X$, we have the following which generalizes [DS, Theorem 3.1] slightly.

Theorem 7.1 (cf. [DS, Theorem 3.1])**.** *For a C^*-correspondence X over a C^*-algebra A, the following conditions are equivalent;*

(i) *A is exact,*
(ii) *$\mathcal{T}_X^{\bar{\gamma}}$ is exact,*
(iii) *$\mathcal{T}_X$ is exact,*
(iv) *$\mathcal{O}_X^{\gamma}$ is exact,*
(v) *$\mathcal{O}_X$ is exact.*

Proof. Suppose that A is exact. By Proposition B.7, $\mathcal{K}(X^{\otimes n})$ is exact for all $n \in \mathbb{N}$. By Proposition 5.14, we can prove inductively that $\bar{\mathcal{B}}_{[0,n]} \subset \mathcal{T}_X^{\bar{\gamma}}$ is exact for all $n \in \mathbb{N}$ because exactness is closed under taking splitting extensions. Thus $\mathcal{T}_X^{\bar{\gamma}}$ is exact because it is an inductive limit of exact C^*-algebras. This proves (i) $\Rightarrow$ (ii). The equivalences (ii) $\iff$ (iii) and (iv) $\iff$ (v) follow from Proposition A.13. Since there exists a surjection $\mathcal{T}_X \to \mathcal{O}_X$, (iii) implies (v). Finally, (v) implies (i) because $\pi_X(A) \subset \mathcal{O}_X$ is isomorphic to A. $\square$

On the nuclearity of $\mathcal{T}_X$, we have the following.

Theorem 7.2. *For a C^*-correspondence X over a C^*-algebra A, the following conditions are equivalent;*

(i) *A is nuclear,*
(ii) *$\mathcal{T}_X^{\bar{\gamma}}$ is nuclear,*
(iii) *$\mathcal{T}_X$ is nuclear.*

Proof. In a similar way to the proof of (i) $\Rightarrow$ (ii) in Theorem 7.1, we can show that (i) implies (ii). The implication (ii) $\Rightarrow$ (i) follows from Proposition 5.15. Finally, Proposition A.13 gives the equivalence (ii) $\iff$ (iii). $\square$

On the nuclearity of $\mathcal{O}_X$, we have the following.

Theorem 7.3. *For a C^*-correspondence X over a C^*-algebra A, the following conditions are equivalent;*

(i) *A/J_X is a nuclear C^*-algebra, and $\pi_X \colon J_X \to \mathcal{B}_{[1,\infty]}$ is a nuclear map,*
(ii) *$\pi_X \colon A \to \mathcal{O}_X^{\gamma}$ is a nuclear map,*
(iii) *$\pi_X \colon A \to \mathcal{O}_X$ is a nuclear map,*
(iv) *$\mathcal{O}_X^{\gamma}$ is nuclear,*
(v) *$\mathcal{O}_X$ is nuclear.*

Proof. The equivalence (i) $\iff$ (ii) is shown by applying Proposition A.6 to the diagram in Proposition 5.17. The equivalence (ii) $\iff$ (iii) follows from Proposition A.12. Obviously (iv) implies (ii). Assume (ii). We see that A/J_X is nuclear from the equivalence (i) $\iff$ (ii). We will prove that the embedding $\mathcal{B}_{[0,n]} \hookrightarrow \mathcal{B}_{[0,\infty]}$ is nuclear for all $n \in \mathbb{N}$ by the induction on n. The case $n = 0$ follows from the condition (ii). Suppose we have shown that $\mathcal{B}_{[0,n]} \hookrightarrow \mathcal{B}_{[0,\infty]}$ is nuclear. Let us set $Y_n = \overline{\operatorname{span}}(t_X(X)\mathcal{B}_{[0,n]})$ and $Y_\infty = \overline{\operatorname{span}}(t_X(X)\mathcal{B}_{[0,\infty]})$. Then by Lemma 5.2, Y_n is a Hilbert $\mathcal{B}_{[0,n]}$-module with $\mathcal{K}(Y_n) \cong \mathcal{B}_{[1,n+1]}$, and Y_∞ is a Hilbert $\mathcal{B}_{[0,\infty]}$-module with $\mathcal{K}(Y_\infty) \cong \mathcal{B}_{[1,\infty]}$. By applying Proposition B.8 to the inclusions $\mathcal{B}_{[0,n]} \hookrightarrow \mathcal{B}_{[0,\infty]}$ and $Y_n \hookrightarrow Y_\infty$, we see that the inclusion $\mathcal{B}_{[1,n+1]} \hookrightarrow \mathcal{B}_{[1,\infty]}$ is nuclear. Now by applying Proposition A.6 to the diagram in Proposition 5.16, we see that $\mathcal{B}_{[0,n+1]} \hookrightarrow \mathcal{B}_{[0,\infty]}$ is nuclear. Hence we have shown that $\mathcal{B}_{[0,n]} \hookrightarrow \mathcal{B}_{[0,\infty]}$ is nuclear for all $n \in \mathbb{N}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$ is dense in $\mathcal{B}_{[0,\infty]}$, we see that the identity map $\mathcal{B}_{[0,\infty]} \to \mathcal{B}_{[0,\infty]}$ is nuclear. Thus $\mathcal{B}_{[0,\infty]}$ is a nuclear C^*-algebra. This shows that (ii) implies (iv). Finally, the equivalence (iv) $\iff$ (v) follows from Proposition A.13. □

We give two sufficient conditions on C^*-correspondences X for $\mathcal{O}_X$ to be nuclear, which may be useful. Both of them easily follows from Theorem 7.3.

Corollary 7.4. *If A is nuclear then $\mathcal{O}_X$ is nuclear.*

Corollary 7.5. *If both the C^*-algebra A/J_X and the $*$-homomorphism $\varphi_X\colon J_X \to \mathcal{K}(X)$ are nuclear, then $\mathcal{O}_X$ is nuclear.*

Remark 7.6. We can prove Corollary 7.4 directly by showing that $\mathcal{O}_X^\gamma$ is nuclear when A is nuclear in a similar way to the proof of (i) $\Rightarrow$ (ii) in Theorem 7.1.

The converses of Corollary 7.4 and Corollary 7.5 are not true as the following example shows. We would like to thank Narutaka Ozawa who gave us this example.

Example 7.7. Let B be a nuclear C^*-algebra, and D be a non-nuclear C^*-subalgebra of B. For an integer n, we define A_n by $A_n = B$ for $n > 0$ and $A_n = D$ for $n \le 0$. We set $A = \bigoplus_{n=-\infty}^{\infty} A_n$. We define an injective endomorphism $\varphi\colon A \to A$ so that $\varphi|_{A_0}\colon A_0 \to A_1$ is a natural embedding and $\varphi|_{A_n}\colon A_n \to A_{n+1}$ is an isomorphism for a non-zero integer n. Since D is not nuclear, the injective endomorphism φ is not nuclear. Let X be the C^*-correspondence over A which is isomorphic to A as Hilbert A-modules, and whose left action $\varphi_X\colon A \to \mathcal{L}(X)$ is defined as the composition of $\varphi\colon A \to A$ and the isomorphism $A \cong \mathcal{K}(X) \subset \mathcal{L}(X)$. Then we have $J_X = A$ and the map $\varphi_X\colon J_X \to \mathcal{K}(X)$ is not nuclear as φ is not. Thus the C^*-correspondence X does not satisfy the assumption of Corollary 7.4 nor Corollary 7.5. However, the C^*-algebra $\mathcal{O}_X$ is nuclear because the fixed point algebra $\mathcal{O}_X^\beta$ is isomorphic to the inductive limit $\varinjlim(A, \varphi) \cong \bigoplus_{n=-\infty}^{\infty} B$, which is nuclear.

A Hilbert A-bimodule X is naturally considered as a C^*-correspondence over A, and the C^*-algebra $\mathcal{O}_X$ is isomorphic to the crossed product $A \rtimes_X \mathbb{Z}$ of A by X defined in [AEE, Definition 2.4] (see [Ka2, Subsection 3.3]). We have a nice characterization of the nuclearity of such a C^*-algebra.

Proposition 7.8. *When a C^*-correspondence X over a C^*-algebra A comes from a Hilbert A-bimodule, the C^*-algebra $\mathcal{O}_X$ is nuclear if and only if A is nuclear.*

Proof. By Proposition 5.18, we see that $\pi_X\colon A \to \mathcal{O}_X^\beta$ is an isomorphism. Hence the conclusion follows from Theorem 7.3, or rather Proposition A.13. $\square$

8. K-groups

The purpose of this section is to obtain the 6-term exact sequence of K-groups, which seems to be useful to compute the K-groups $K_0(\mathcal{O}_X)$ and $K_1(\mathcal{O}_X)$ of $\mathcal{O}_X$. Mainly we follow the arguments in [P, Section 4]. There, Pimsner used KK-theory to obtain his 6-term exact sequence. For this reason, he assumed the separability of the C^*-algebras involved. Here, we work directly with K-theory instead of using KK-theory, and obtain the 6-term exact sequence without the assumption of separability.

For a C^*-algebra A, we denote by $K_*(A)$ the K-group $K_0(A) \oplus K_1(A)$ of A which has a $\mathbb{Z}/2\mathbb{Z}$-grading. By maps between K-groups, we mean group homomorphisms which preserve the grading. Thus for C^*-algebras A and B, considering maps between K-groups $K_*(A) \to K_*(B)$ is same as considering two homomorphisms $K_0(A) \to K_0(B)$ and $K_1(A) \to K_1(B)$. For a $*$-homomorphism $\rho\colon A \to B$, we denote by ρ_* the map $K_*(A) \to K_*(B)$ induced by ρ.

Fix a C^*-correspondence X over a C^*-algebra A. Since we have $\mathcal{T}_X \cong C^*(\varphi_\infty, \tau_\infty)$ by Proposition 6.5, there exists an embedding $j\colon \mathcal{K}(\mathcal{F}(X)J_X) \to \mathcal{T}_X$ by Proposition 4.6. Since $C^*(\varphi, \tau) \cong \mathcal{O}_X$ by Proposition 6.5, we have the following short exact sequence;

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(X)J_X) \xrightarrow{\ j\ } \mathcal{T}_X \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

The following two propositions enable us to compute the K-groups of $\mathcal{K}(\mathcal{F}(X)J_X)$ and $\mathcal{T}_X$.

Proposition 8.1. *The $*$-homomorphism $\varphi_0\colon J_X \to \mathcal{K}(\mathcal{F}(X)J_X)$ induces an isomorphism $(\varphi_0)_*\colon K_*(J_X) \to K_*(\mathcal{K}(\mathcal{F}(X)J_X))$.*

Proof. The $*$-homomorphism $\varphi_0\colon J_X \to \mathcal{K}(\mathcal{F}(X)J_X)$ is an isomorphism onto the C^*-subalgebra $\mathcal{K}(X^{\otimes 0}J_X)$ of $\mathcal{K}(\mathcal{F}(X)J_X)$. Since $X^{\otimes 0}J_X$ is a full Hilbert J_X-submodule of $\mathcal{F}(X)J_X$, $\mathcal{K}(X^{\otimes 0}J_X)$ is a hereditary and full C^*-subalgebra of $\mathcal{K}(\mathcal{F}(X)J_X)$. Hence $(\varphi_0)_*$ is an isomorphism by Proposition B.5. $\square$

Proposition 8.2. *The $*$-homomorphism $\bar{\pi}_X\colon A \to \mathcal{T}_X$ induces an isomorphism $(\bar{\pi}_X)_*\colon K_*(A) \to K_*(\mathcal{T}_X)$.*

Proof. See Appendix C. $\square$

Next, we will compute $j_*\colon K_*(\mathcal{K}(\mathcal{F}(X)J_X)) \to K_*(\mathcal{T}_X)$.

Definition 8.3. We denote by $\iota\colon J_X \hookrightarrow A$ the natural embedding. We define a map $[X]\colon K_*(J_X) \to K_*(A)$ by the composition of the map $(\varphi_X)_*\colon K_*(J_X) \to K_*(\mathcal{K}(X))$ induced by the restriction of φ_X to J_X and the map $X_*\colon K_*(\mathcal{K}(X)) \to K_*(A)$ induced by the Hilbert A-module X as in Remark B.4.

The map $[X]\colon K_*(J_X) \to K_*(A)$ is same as the map induced by the element $(X, \varphi_X, 0)$ of $KK(J_X, A)$. When a C^*-correspondence X is defined from an injective $*$-homomorphism $\varphi\colon A \to A$, we have $J_X = A$ and $[X] = \varphi_*$. For the notation in the proof of the next lemma, consult Appendix B.

Lemma 8.4. *The composition of the two maps $[X]\colon K_*(J_X) \to K_*(A)$ and $(\bar{\pi}_X)_*\colon K_*(A) \to K_*(\mathcal{T}_X)$ coincides with $(\psi_{\bar{t}_X} \circ \varphi_X)_*$.*

Proof. Let $M_2(\mathcal{T}_X)$ be the C^*-algebra of two-by-two matrices with entries in $\mathcal{T}_X$. For $i, j \in \{0,1\}$, we denote by ι_{ij} the natural embedding $\mathcal{T}_X \to M_2(\mathcal{T}_X)$ onto the i,j-component. By the definition of K-groups, $(\iota_{00})_* = (\iota_{11})_*$ is an isomorphism.

From the maps $\bar{\pi}_X \colon A \to \mathcal{T}_X$ and $\bar{t}_X \colon X \to \mathcal{T}_X$, we get a $*$-homomorphism $\rho \colon D_X \to M_2(\mathcal{T}_X)$ such that $\rho \circ \iota_A = \iota_{11} \circ \bar{\pi}_X$ and $\rho \circ \iota_X = \iota_{01} \circ \bar{t}_X$. We have $\rho \circ \iota_{\mathcal{K}(X)} = \iota_{00} \circ \psi_{\bar{t}_X}$. Since X_* is defined as $(\iota_A)_*^{-1} \circ (\iota_{\mathcal{K}(X)})_*$, we have

$$\begin{aligned}(\bar{\pi}_X)_* \circ X_* &= (\bar{\pi}_X)_* \circ (\iota_A)_*^{-1} \circ (\iota_{\mathcal{K}(X)})_* \\ &= (\iota_{11})_*^{-1} \circ \rho_* \circ (\iota_{\mathcal{K}(X)})_* \\ &= (\iota_{11})_*^{-1} \circ (\iota_{00})_* \circ (\psi_{\bar{t}_X})_* \\ &= (\psi_{\bar{t}_X})_*.\end{aligned}$$

Hence we get

$$(\bar{\pi}_X)_* \circ [X] = (\bar{\pi}_X)_* \circ X_* \circ (\varphi_X)_* = (\psi_{\bar{t}_X})_* \circ (\varphi_X)_* = (\psi_{\bar{t}_X} \circ \varphi_X)_*.$$

We are done. $\square$

Lemma 8.5. *The $*$-homomorphism $\bar{\pi}_X \circ \iota \colon J_X \to \mathcal{T}_X$ is the sum of the two $*$-homomorphisms $\psi_{\bar{t}_X} \circ \varphi_X$ and $j \circ \varphi_0$.*

Proof. If we identify $\mathcal{T}_X$ and $C^*(\varphi_\infty, \tau_\infty)$, this follows from Proposition 4.4. $\square$

By the two lemma above, the map $j_* \colon K_*(\mathcal{K}(\mathcal{F}(X)J_X)) \to K_*(\mathcal{T}_X)$ is same as the map $\iota_* - [X] \colon K_*(J_X) \to K_*(A)$ modulo the isomorphisms $(\varphi_0)_* \colon K_*(J_X) \to K_*(\mathcal{K}(\mathcal{F}(X)J_X))$ and $(\bar{\pi}_X)_* \colon K_*(A) \to K_*(\mathcal{T}_X)$:

$$\begin{array}{ccc} K_*(\mathcal{K}(\mathcal{F}(X)J_X)) & \xrightarrow[j_*]{} & K_*(\mathcal{T}_X) \\ \uparrow{\scriptstyle (\varphi_0)_*} & & \uparrow{\scriptstyle (\bar{\pi}_X)_*} \\ K_*(J_X) & \xrightarrow[\iota_* - [X]]{} & K_*(A). \end{array}$$

Thus by rewriting the 6-term exact sequence of K-groups obtained from the short exact sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(X)J_X) \xrightarrow{\ j\ } \mathcal{T}_X \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we get the following.

Theorem 8.6 (cf. [P, Theorem 4.9])**.** *For a C^*-correspondence X over a C^*-algebra A, we have the following exact sequence;*

$$\begin{array}{ccccc} K_0(J_X) & \xrightarrow[\iota_* - [X]]{} & K_0(A) & \xrightarrow[(\pi_X)_*]{} & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \xleftarrow{(\pi_X)_*} & K_1(A) & \xleftarrow{\iota_* - [X]} & K_1(J_X). \end{array}$$

For a C^*-correspondence X over a C^*-algebra A and an ideal J of A satisfying $\varphi_X(J) \subset \mathcal{K}(X)$, the relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ is defined as the quotient $C^*(\varphi_\infty, \tau_\infty)/\mathcal{K}(\mathcal{F}(X)J)$ ([MS, Definition 2.18]). Thus we can prove the following statement in the same way as the proof of Theorem 8.6.

Proposition 8.7. *Let X be a C^*-correspondence over a C^*-algebra A, and J be an ideal of A with $\varphi_X(J) \subset \mathcal{K}(X)$. Then we have the following exact sequence;*

$$
\begin{array}{ccccc}
K_0(J) & \xrightarrow[\iota_* - [X,J]]{} & K_0(A) & \xrightarrow[\pi_*]{} & K_0(\mathcal{O}(J,X)) \\
\uparrow & & & & \downarrow \\
K_1(\mathcal{O}(J,X)) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{\iota_* - [X,J]} & K_1(J),
\end{array}
$$

where $\iota\colon J \hookrightarrow A$ is the embedding, $\pi\colon A \to \mathcal{O}(J,X)$ is the natural $*$-homomorphism, and $[X,J]\colon K_*(J) \to K_*(A)$ is defined by $[X,J] = X_* \circ (\varphi_X|_J)_*$.

It is not difficult to see that the two $*$-homomorphisms in Proposition 8.1 and Proposition 8.2 induce KK-equivalences between J_X and $\mathcal{K}(\mathcal{F}(X)J_X)$ and between A and $\mathcal{T}_X$ when the involving C^*-algebras are separable. Hence by applying "two among three principle" to the short exact sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(X)J_X) \xrightarrow{j} \mathcal{T}_X \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we get the following.

Proposition 8.8. *Let X be a separable C^*-correspondence over a separable nuclear C^*-algebra A. If A and J_X satisfy the Universal Coefficient Theorem of* [RS]*, then so does $\mathcal{O}_X$.*

Appendix A. On nuclear maps

In Appendices A and B, we gather the results on nuclear maps and linking algebras. We use these results in Sections 7 and 8. Most of them should be known among the specialists. Some results in this appendix hold with less assumption.

Definition A.1. For C^*-algebras A and D, we denote by $A \otimes_{\min} D$ (resp. $A \otimes_{\max} D$) the *minimal (resp. maximal) tensor product* of A and D, and by $A \ominus D$ the kernel of the natural surjection $\pi_{A,D}\colon A \otimes_{\max} D \to A \otimes_{\min} D$.

Definition A.2. For a $*$-homomorphism $\varphi\colon A \to B$, we can define $*$-homomorphisms $\varphi \otimes_{\min} \mathrm{id}_D\colon A \otimes_{\min} D \to B \otimes_{\min} D$ and $\varphi \otimes_{\max} \mathrm{id}_D\colon A \otimes_{\max} D \to B \otimes_{\max} D$ such that $\varphi \otimes_{\min} \mathrm{id}_D(a \otimes d) = \varphi \otimes_{\max} \mathrm{id}_D(a \otimes d) = \varphi(a) \otimes d$ for $a \in A$ and $d \in D$. Since we have the commutative diagram;

$$
\begin{tikzcd}
A \otimes_{\max} D \arrow[r, "\varphi \otimes_{\max} \mathrm{id}_D"] \arrow[d, "\pi_{A,D}"] & B \otimes_{\max} D \arrow[d, "\pi_{B,D}"] \\
A \otimes_{\min} D \arrow[r, "\varphi \otimes_{\min} \mathrm{id}_D"] & B \otimes_{\min} D,
\end{tikzcd}
$$

the restriction of $\varphi \otimes_{\max} \mathrm{id}_D$ to $A \ominus D \subset A \otimes_{\max} D$ induces a $*$-homomorphism $\varphi \ominus \mathrm{id}_D\colon A \ominus D \to B \ominus D$.

Definition A.3. A $*$-homomorphism $\varphi\colon A \to B$ is said to be *nuclear* if for all C^*-algebra D, the $*$-homomorphism $\varphi \otimes_{\max} \mathrm{id}_D\colon A \otimes_{\max} D \to B \otimes_{\max} D$ factors through

the surjection $\pi_{A,D}\colon A\otimes_{\max} D \to A\otimes_{\min} D$;

$$\begin{tikzcd}
A\otimes_{\max} D \arrow[r, "\varphi\otimes_{\max}\mathrm{id}_D"] \arrow[d, "\pi_{A,D}"] & B\otimes_{\max} D \arrow[d, "\pi_{B,D}"] \\
A\otimes_{\min} D \arrow[r, "\varphi\otimes_{\min}\mathrm{id}_D"] \arrow[ur, dotted] & B\otimes_{\min} D.
\end{tikzcd}$$

A C^*-algebra A is said to be *nuclear* if $\mathrm{id}_A\colon A\to A$ is a nuclear map.

In other words, a $*$-homomorphism $\varphi\colon A\to B$ is nuclear if and only if $\varphi\ominus\mathrm{id}_D = 0$ for all C^*-algebra D, and a C^*-algebra A is nuclear if and only if $A\ominus D = 0$ for all C^*-algebra D.

Remark A.4. A $*$-homomorphism is nuclear if and only if it has the completely positive approximation property (see [W]).

Lemma A.5. *Let*

$$0 \longrightarrow I \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

be a short exact sequence of C^*-algebras, and D be a C^*-algebra. Then the following sequence is exact;

$$0 \longrightarrow I\ominus D \xrightarrow{\iota\ominus\mathrm{id}_D} A\ominus D \xrightarrow{\pi\ominus\mathrm{id}_D} B\ominus D.$$

If there exists an injective nuclear $*$-homomorphism $A\to A'$ for some C^*-algebra A', then $\pi\ominus\mathrm{id}_D$ is surjective.

Proof. The former statement follows from the fact that maximal tensor products preserve short exact sequences. If there exists an injective nuclear $*$-homomorphism $A\to A'$ for some C^*-algebra A', then A is exact by [W, Proposition 7.2]. Since exact C^*-algebras have Property C [Ki], the sequence

$$0 \longrightarrow I\otimes_{\min} D \xrightarrow{\iota\otimes_{\min}\mathrm{id}_D} A\otimes_{\min} D \xrightarrow{\pi\otimes_{\min}\mathrm{id}_D} B\otimes_{\min} D \longrightarrow 0$$

is exact (see Proposition 5.2 and Remarks 9.5.2 in [W]). Hence the conclusion follows from 3×3-lemma. $\square$

Proposition A.6. *Suppose that we have a following commutative diagram with exact rows;*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "\varphi_0"] & A \arrow[r, "\pi"] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I' \arrow[r, "\iota'"] & A' \arrow[r, "\pi'"] & B \arrow[r] & 0.
\end{tikzcd}$$

Suppose also that φ is injective. Then φ is nuclear if and only if both B and φ_0 are nuclear.

Proof. Take a C^*-algebra D. By Lemma A.5 we have the following commutative diagram with exact rows;

$$\begin{array}{ccccccc}
0 & \longrightarrow & I \ominus D & \xrightarrow{\iota\ominus\mathrm{id}_D} & A \ominus D & \xrightarrow{\pi\ominus\mathrm{id}_D} & B \ominus D \\
 & & \Big\downarrow{\scriptstyle \varphi_0\ominus\mathrm{id}_D} & & \Big\downarrow{\scriptstyle \varphi\ominus\mathrm{id}_D} & & \Big\| \\
0 & \longrightarrow & I' \ominus D & \xrightarrow{\iota'\ominus\mathrm{id}_D} & A' \ominus D & \xrightarrow{\pi'\ominus\mathrm{id}_D} & B \ominus D.
\end{array}$$

Suppose that φ is nuclear. By Lemma A.5, the $*$-homomorphism $\pi\ominus\mathrm{id}_D$ is surjective. Hence we have $B \ominus D = 0$ for all C^*-algebra D. We also have $\varphi_0 \ominus \mathrm{id}_D = 0$ for all C^*-algebra D by the diagram above. Thus both B and φ_0 are nuclear. Conversely assume that both B and φ_0 are nuclear. Then we have $\varphi\ominus\mathrm{id}_D = 0$ for all C^*-algebra D by the diagram above. Therefore φ is nuclear. We are done. □

Proposition A.7. *Let A, B be C^*-algebras, and A_0, B_0 be C^*-subalgebras of A and B, respectively. Let $\varphi\colon A \to B$ be a $*$-homomorphism with $\varphi(A_0) \subset B_0$. Let $\varphi_0\colon A_0 \to B_0$ be the restriction of φ. When B_0 is a hereditary C^*-subalgebra of B, the nuclearity of φ implies the nuclearity of φ_0.*

Proof. When φ is nuclear, its restriction $\varphi'\colon A_0 \to B$ is also nuclear. Hence for any C^*-algebra D, the map $\varphi' \ominus \mathrm{id}_D\colon A_0 \ominus D \to B \ominus D$ is 0. Since B_0 is a hereditary C^*-subalgebra of B, we see that the inclusion $\iota\colon B_0 \hookrightarrow B$ induces an injective $*$-homomorphism $\iota \otimes_{\max} \mathrm{id}_D\colon B_0 \otimes_{\max} D \to B \otimes_{\max} D$ by [L1, Theorem 3.3]. Hence the $*$-homomorphism $\iota \ominus \mathrm{id}_D\colon B_0 \ominus D \to B \ominus D$ is also injective. This shows that $\varphi_0 \ominus \mathrm{id}_D\colon A_0 \ominus D \to B_0 \ominus D$ is 0 for all C^*-algebra D. Thus φ_0 is injective. □

The following complements the proposition above.

Proposition A.8. *With the same notation in Proposition A.7, when A_0 is a hereditary and full C^*-subalgebra of A, the nuclearity of φ_0 implies the nuclearity of φ.*

Proof. Take a C^*-algebra D. Since A_0 is a hereditary and full C^*-subalgebra of A, $A_0 \otimes_{\max} D$ is a hereditary and full C^*-subalgebra of $A \otimes_{\max} D$. Hence $A_0 \ominus D = (A_0 \otimes_{\max} D) \cap (A \ominus D)$ is also hereditary and full in $A \ominus D$. When φ_0 is nuclear, the $*$-homomorphism $\varphi \otimes_{\max} \mathrm{id}_D\colon A \otimes_{\max} D \to B \otimes_{\max} D$ vanishes on $A_0 \ominus D$. Thus $\varphi \otimes_{\max} \mathrm{id}_D$ vanishes on $A \ominus D$. This shows that φ is nuclear. □

The following is an immidiate consequence of Proposition A.7 and Proposition A.8.

Corollary A.9. *A hereditary and full C^*-subalgebra A_0 of a C^*-algebra A is nuclear if and only if A is nuclear.*

We also have the following.

Proposition A.10. *A hereditary and full C^*-subalgebra A_0 of a C^*-algebra A is exact if and only if A is exact.*

Proof. Since a C^*-subalgebra of an exact C^*-algebra is exact, A_0 is exact if A is exact. Suppose that A_0 is exact. Take a short exact sequence of C^*-algebras;

$$0 \longrightarrow I \xrightarrow{\ \iota\ } B \xrightarrow{\ \pi\ } D \longrightarrow 0.$$

All we have to do is to prove $\ker(\pi \otimes_{\min} \mathrm{id}_A) = I \otimes_{\min} A$. Since A_0 is full and hereditary in A, $B \otimes_{\min} A_0$ is full and hereditary in $B \otimes_{\min} A$. Thus $\ker(\pi \otimes_{\min} \mathrm{id}_A)$ is generated by its intersection with $B \otimes_{\min} A_0$, which is $I \otimes_{\min} A_0$ by the exactness of A_0. Hence we get $\ker(\pi \otimes_{\min} \mathrm{id}_A) = I \otimes_{\min} A$. We are done. $\square$

Remark A.11. We can prove Proposition A.10 by using Proposition A.8 together with the deep fact that a C^*-algebra is exact if and only if its one (or all) faithful representation is nuclear due to Kirchberg [Ki]. We can also prove Proposition A.10 in a similar way to the proof of Proposition B.3.

The above investigation of hereditary C^*-subalgebras can be extended to other classes of C^*-subalgebras. In Section 7, we just need the following two results.

Proposition A.12. *Let $\alpha\colon G \curvearrowright A$ be an action of a compact group G on a C^*-algebra A. Let $\varphi\colon D \to A$ be a $*$-homomorphism whose image is contained in the fixed point algebra A^α of α. Then the restriction $\varphi_0\colon D \to A^\alpha$ is nuclear if and only if φ is nuclear.*

Proof. Similar as the proof of Proposition A.7. $\square$

Proposition A.13. *Let $\alpha\colon G \curvearrowright A$ be an action of a compact group G on a C^*-algebra A. Then A is nuclear or exact if and only if the fixed point algebra A^α is also.*

Proof. For nuclearity, it was proved in [DLRZ, Proposition 2]. It was pointed out by Narutaka Ozawa that the technique in [DLRZ] works for exactness. We will sketch his argument.

When A is exact, A^α is exact. Assume that A^α is exact. Take a short exact sequence of C^*-algebras;

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\ \pi\ } D \longrightarrow 0.$$

Let us take a positive element x of $\ker(\pi \otimes_{\min} \mathrm{id}_A)$. To derive a contradiction, we assume $x \notin I \otimes_{\min} A$. Then we can find a state φ of $B \otimes_{\min} A$ such that φ vanishes on $I \otimes_{\min} A$ and $\varphi(x) > 0$. We set $x_0 = \int_G \mathrm{id}_B \otimes_{\min} \alpha_z(x) dz$ where dz is the normalized Haar measure of G. Then we see $x_0 \in B \otimes_{\min} A^\alpha$. We have

$$\begin{aligned}(\pi \otimes_{\min} \mathrm{id}_{A^\alpha})(x_0) &= \int_G \pi \otimes_{\min} \mathrm{id}_A \big(\mathrm{id}_B \otimes_{\min} \alpha_z(x)\big) dz \\ &= \int_G \mathrm{id}_D \otimes_{\min} \alpha_z\big(\pi \otimes_{\min} \mathrm{id}_A(x)\big) dz = 0.\end{aligned}$$

Since A^α is exact, we have $x_0 \in I \otimes_{\min} A^\alpha$. This leads a contradiction as

$$0 = \varphi(x_0) = \int_G \varphi(\mathrm{id}_B \otimes_{\min} \alpha_z(x)) dz > 0.$$

Therefore we have $x \in I \otimes_{\min} A$ for all positive element x of $\ker(\pi \otimes_{\min} \mathrm{id}_A)$. Thus we have shown $\ker(\pi \otimes_{\min} \mathrm{id}_A) = I \otimes_{\min} A$. This implies that A is exact. $\square$

Appendix B. On linking algebras

Definition B.1. Let A be a C^*-algebra and X be a Hilbert A-module. The C^*-algebra $\mathcal{K}(X \oplus A)$ is called the *linking algebra* of X, and denoted by D_X.

Since $\mathcal{K}(A,X) \cong X$ and $\mathcal{K}(A) \cong A$ naturally, we have the following matrix representation of D_X;

$$D_X = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix},$$

where $\widetilde{X} = \mathcal{K}(X,A)$ is the dual left Hilbert A-module of X. The natural embeddings are denoted by

$$\iota_{\mathcal{K}(X)}\colon \mathcal{K}(X) \hookrightarrow D_X, \quad \iota_X\colon X \hookrightarrow D_X, \quad \text{and} \quad \iota_A\colon A \hookrightarrow D_X.$$

Both maps ι_A and $\iota_{\mathcal{K}(X)}$ are injective $*$-homomorphisms onto corners of D_X. The C^*-subalgebra A of D_X is always full, but $\mathcal{K}(X)$ is full in D_X only in the case that X is a full Hilbert A-module.

Lemma B.2. *Let A be a C^*-algebra and X be a Hilbert A-module. For separable subsets $A_0 \subset A$ and $X_0 \subset X$, there exist a separable C^*-subalgebra $A_\infty \subset A$ containing A_0 and a separable closed subspace X_∞ of X containing X_0 such that X_∞ is a Hilbert A_∞-module by restricting the operations of X.*

Proof. Let A_1 be the C^*-algebra generated by $A_0 + \langle X_0, X_0\rangle_X$. We set $X_1 = \overline{\mathrm{span}}(X_0 + X_0A_0)$ which is a closed subspace of X. We inductively define families of separable C^*-subalgebras $\{A_n\}_{n=1}^\infty$ of A and separable closed subspaces $\{X_n\}_{n=1}^\infty$ of X so that A_{n+1} is a C^*-algebra generated by $A_n + \langle X_n, X_n\rangle_X$, and that $X_{n+1} = \overline{\mathrm{span}}(X_n + X_nA_n)$. We set $A_\infty = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ and $X_\infty = \overline{\bigcup_{n\in\mathbb{N}} X_n}$. Then A_∞ is a separable C^*-subalgebra of A containing A_0, and X_∞ is a separable closed subspace of X containing X_0. By the construction, we have $X_\infty A_\infty \subset X_\infty$ and $\langle X_\infty, X_\infty\rangle_X \subset A_\infty$. Hence X_∞ is a Hilbert A_∞-module. □

Proposition B.3. *For a C^*-algebra A and a Hilbert A-module X, the inclusion $\iota_A\colon A \to D_X$ induces an isomorphism on the K-groups.*

Proof. When both A and X are separable, [B, Corollary 2.6] gives us an isometry v in the multiplier algebra $\mathcal{M}(D_X \otimes_{\min} \mathbb{K})$ of $D_X \otimes_{\min} \mathbb{K}$ such that $\Phi\colon D_X \otimes_{\min} \mathbb{K} \ni x \mapsto vxv^* \in A \otimes_{\min} \mathbb{K}$ is an isomorphism, where $\mathbb{K}$ is the C^*-algebra of the compact operators on the infinite-dimensional separable Hilbert space. Since the composition of the isomorphism Φ and the inclusion $\iota_A \otimes_{\min} \mathrm{id}_{\mathbb{K}}\colon A \otimes_{\min} \mathbb{K} \to D_X \otimes_{\min} \mathbb{K}$ induces an identity on the K-groups of $D_X \otimes_{\min} \mathbb{K}$ (see, for example, [HR, Lemma 4.6.2]), the inclusion $\iota_A \otimes_{\min} \mathrm{id}_{\mathbb{K}}$ induces an isomorphism on the K-groups. Hence the inclusion $\iota_A\colon A \to D_X$ also induces an isomorphism on the K-groups.

Now let A be a general C^*-algebra and X be a general Hilbert A-module. By Lemma B.2, the set of the pairs (A_λ, X_λ) consisting of separable C^*-subalgebras A_λ of A and separable closed subspaces X_λ of X such that X_λ are Hilbert A_λ-modules is upward directed with respect to the inclusions, and satisfies $A = \bigcup_\lambda A_\lambda$, $X = \bigcup_\lambda X_\lambda$. We have $A \cong \varinjlim A_\lambda$ and $D_X \cong \varinjlim D_{X_\lambda}$. By the first part of this proof, the inclusion $\iota_{A_\lambda}\colon A_\lambda \to D_{X_\lambda}$ induces an isomorphism on the K-groups for all λ. Thus the inclusion $\iota_A\colon A \to D_X$ also induces an isomorphism on the K-groups. □

Remark B.4. Let A be a C^*-algebra and X be a Hilbert A-module. By Proposition B.3, we can define a map $X_*\colon K_*(\mathcal{K}(X)) \to K_*(A)$ by the composition of the map $(\iota_{\mathcal{K}(X)})_*\colon K_*(\mathcal{K}(X)) \to K_*(D_X)$ and the inverse of the isomorphism $(\iota_A)_*\colon K_*(A) \to K_*(D_X)$. This map is the same map as the one defined in [E, Definition 5.1].

Proposition B.5. *Let A, B be C^*-algebras, and $\iota\colon A \to B$ be an injective $*$-homomorphism onto a hereditary and full C^*-subalgebra of B. Then ι_* is an isomorphism from $K_*(A)$ to $K_*(B)$.*

Proof. The proof goes the same way as the proof of [B, Corollary 2.10] with the help of Proposition B.3. □

Remark B.6. Let A, B be strongly Morita equivalent C^*-algebras. Then there exists a C^*-algebra D which contains A and B as full and hereditary C^*-subalgebras. Hence we see that the K-groups of A and B are isomorphic by Proposition B.5, and that A is nuclear or exact if and only if B is also by Corollary A.9 and Proposition A.10.

We use the two propositions below in Section 7.

Proposition B.7. *Let A be a C^*-algebra and X be a Hilbert A-module. If A is nuclear or exact, then $\mathcal{K}(X)$ is also.*

Proof. Since A is a hereditary and full C^*-subalgebra of D_X, if A is nuclear or exact then D_X is also by Corollary A.9 and Proposition A.10. Now the conclusion follows from the fact that $\mathcal{K}(X)$ is a hereditary C^*-subalgebra of D_X. □

Proposition B.8. *Let A and B be C^*-algebras, X be a Hilbert A-module, and Y be a Hilbert B-module. Let $\pi\colon A \to B$ be a $*$-homomorphism and $t\colon X \to Y$ be a linear map satisfying $\langle t(\xi), t(\eta)\rangle_Y = \pi(\langle \xi, \eta\rangle_X)$ for $\xi, \eta \in X$. We can define a $*$-homomorphism $\psi_t\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ by $\psi_t(\theta_{\xi,\eta}) = \theta_{t(\xi),t(\eta)}$ for $\xi, \eta \in X$. Then the nuclearity of π implies the nuclearity of ψ_t.*

Proof. For the well-definedness of ψ_t, see [KPW, Lemma 2.2]. We can define a $*$-homomorphism $\rho\colon D_X \to D_Y$ so that $\rho \circ \iota_A = \iota_B \circ \pi$, $\rho \circ \iota_X = \iota_Y \circ t$ and $\rho \circ \iota_{\mathcal{K}(X)} = \iota_{\mathcal{K}(Y)} \circ \psi_t$. Since A is a hereditary and full C^*-subalgebra of D_X, the nuclearity of π implies the nuclearity of ρ by Proposition A.8. Since $\mathcal{K}(Y)$ is a hereditary C^*-subalgebra of D_Y, the nuclearity of ρ implies the nuclearity of ψ_t by Proposition A.7. We are done. □

Appendix C. A proof of Proposition 8.2

In this appendix, we give a K-theoretical proof of Proposition 8.2. In [P, Theorem 4.4], Pimsner used KK-theory to prove this proposition under some hypotheses, one of which is that both A and X are separable. What we will do here is to get rid of KK-theory from the proof of [P, Theorem 4.4] so that we can prove this proposition without the assumption of separability. We first prepare some notation and results which we will need.

Definition C.1. For a C^*-algebra A, we define $SA = C_0((0,1), A)$, which we often consider as a set of functions in $C_0((-1,1), A)$ vanishing on $(-1, 0]$. For a $*$-homomorphism $\varphi\colon A \to B$, we denote by $S\varphi\colon SA \to SB$ the $*$-homomorphism defined by $S\varphi(f)(s) = \varphi(f(s))$ for $f \in SA$ and $s \in (0,1)$.

Definition C.2. For a C^*-algebra A and an ideal I of A, we define a C^*-algebra $D(I, A)$ by

$$D(I, A) = \{f \in C_0((-1,1), A) \mid f(s) - f(-s) \in I \text{ for all } s \in (-1,1)\}.$$

We denote by ι the natural embedding $SI \to D(I, A)$.

Lemma C.3. *The $*$-homomorphism $\iota\colon SI \to D(I,A)$ induces an isomorphism $\iota_*\colon K_*(SI) \to K_*(D(I,A))$.*

Proof. Let us define a $*$-homomorphism $\pi\colon D(I,A) \to C_0((-1,0],A)$ by the restriction. Then π is surjective and its kernel is SI. Hence we have the following short exact sequence

$$0 \longrightarrow SI \xrightarrow{\ \iota\ } D(I,A) \xrightarrow{\ \pi\ } C_0((-1,0],A) \longrightarrow 0.$$

The conclusion follows from the 6-term exact sequence of K-groups associated with this short exact sequence together with the fact that $K_*(C_0((-1,0],A)) = 0$. $\square$

Definition C.4. Let A, B be C^*-algebras, and I be an ideals of A. For two $*$-homomorphisms $\rho_+, \rho_-\colon B \to A$ such that $\rho_+(b) - \rho_-(b) \in I$ for all $b \in B$, we define a $*$-homomorphisms $\rho\colon SB \to D(I,A)$ by

$$\rho(f)(s) = \begin{cases} \rho_+(f(s)) & \text{if } s \geq 0 \\ \rho_-(f(-s)) & \text{if } s \leq 0, \end{cases}$$

for $f \in SB$.

Lemma C.5. *When $\rho_+ = \rho_-$, the $*$-homomorphism $\rho\colon SB \to D(I,A)$ in Definition C.4 induces 0 on K-groups.*

Proof. When $\rho_+ = \rho_-$, the $*$-homomorphism ρ factors through the $*$-homomorphism $\sigma\colon C_0([0,1),A) \to D(I,A)$ defined by

$$\sigma(f)(s) = \begin{cases} f(s) & \text{if } s \geq 0 \\ f(-s) & \text{if } s \leq 0, \end{cases}$$

for $f \in C_0([0,1),A)$. Since $K_*(C_0([0,1),A)) = 0$, we have $\rho_* = 0$. $\square$

Lemma C.6. *For $j = 1,2$, let A_j be a C^*-algebra, and I_j be an ideal of A_j. For a $*$-homomorphism $\varphi\colon A_1 \to A_2$ with $\varphi(I_1) \subset I_2$, we can define a $*$-homomorphism $D\varphi\colon D(I_1,A_1) \to D(I_2,A_2)$ by $D\varphi(f)(s) = \varphi(f(s))$, and we get a commutative diagram;*

$$\begin{array}{ccc} SI_1 & \xrightarrow[S\varphi]{} & SI_2 \\ \Big\downarrow{\scriptstyle \iota_1} & & \Big\downarrow{\scriptstyle \iota_2} \\ D(I_1,A_1) & \xrightarrow[D\varphi]{} & D(I_2,A_2). \end{array}$$

Proof. Straightforward. $\square$

We go back to the proof of Proposition 8.2. We first treat the case that the C^*-correspondence X is non-degenerate. Let us take a C^*-algebra A and a non-degenerate C^*-correspondence X.

Let $(\varphi_\infty, \tau_\infty)$ be the Fock representation of X on $\mathcal{L}(\mathcal{F}(X))$. We denote by $\rho_+\colon \mathcal{T}_X \to \mathcal{L}(\mathcal{F}(X))$ the $*$-homomorphism such that $\rho_+ \circ \bar{\pi}_X = \varphi_\infty$ and $\rho_+ \circ \bar{t}_X = \tau_\infty$. We define a $*$-homomorphism $\varphi_\infty^-\colon A \to \mathcal{L}(\mathcal{F}(X))$ and a linear map $\tau_\infty^-\colon X \to \mathcal{L}(\mathcal{F}(X))$ by

$$\varphi_\infty^-(a) = \sum_{m=1}^{\infty} \varphi_m(a), \qquad \tau_\infty^-(\xi) = \sum_{m=1}^{\infty} \tau_m^1(\xi).$$

Similarly as the proof of Proposition 4.3, we see that $(\varphi_\infty^-, \tau_\infty^-)$ is a representation of X. Hence there exists a $*$-homomorphism $\rho_-\colon \mathcal{T}_X \to \mathcal{L}(\mathcal{F}(X))$ such that $\rho_- \circ \bar{\pi}_X = \varphi_\infty^-$ and $\rho_- \circ \bar{t}_X = \tau_\infty^-$.

Lemma C.7 ([P, Lemma 4.2])**.** *For every $x \in \mathcal{T}_X$, we have $\rho_+(x) - \rho_-(x) \in \mathcal{K}(\mathcal{F}(X))$.*

Proof. Since $\mathcal{T}_X$ is generated by the image of the two maps $\bar{\pi}_X$ and $\bar{t}_X$, it suffices to show this lemma when $x \in \mathcal{T}_X$ is in the image of these maps. For $a \in A$, we have

$$\rho_+(\bar{\pi}_X(a)) - \rho_-(\bar{\pi}_X(a)) = \varphi_0(a) \in \mathcal{K}(\mathcal{F}(X)),$$

and for $\xi \in X$, we have

$$\rho_+(\bar{t}_X(\xi)) - \rho_-(\bar{t}_X(\xi)) = \tau_0^1(\xi) \in \mathcal{K}(\mathcal{F}(X)).$$

We are done. □

Let us set $D = D\big(\mathcal{K}(\mathcal{F}(X)), \mathcal{L}(\mathcal{F}(X))\big)$. By Lemma C.7, we can define a $*$-homomorphism $\rho\colon S\mathcal{T}_X \to D$ by

$$\rho(f)(s) = \begin{cases} \rho_+(f(s)) & \text{if } s \geq 0 \\ \rho_-(f(-s)) & \text{if } s \leq 0. \end{cases}$$

Lemma C.8. *The $*$-homomorphism $S\varphi_0\colon SA \to D$ induces an isomorphism on the K-groups.*

Proof. This follows from the fact that $\varphi_0\colon A \to \mathcal{K}(\mathcal{F}(X))$ is an injection onto a hereditary and full C^*-subalgebra of $\mathcal{K}(\mathcal{F}(X))$ with the help of Proposition B.5 and Lemma C.3. □

Proposition C.9. *The composition of $S\bar{\pi}_X\colon SA \to S\mathcal{T}_X$ and $\rho\colon S\mathcal{T}_X \to D$ induces an isomorphism on the K-groups.*

Proof. Since we have $\rho_+ \circ \bar{\pi}_X = \varphi_0 + \rho_- \circ \bar{\pi}_X$, we can see that the composition $\rho \circ S\bar{\pi}_X$ induces the same map as $S\varphi_0$ with the help of Lemma C.5. Hence the proof completes by Lemma C.8. □

Proposition C.9 implies that ρ_* is "the left inverse" of the map $(S\bar{\pi}_X)_*\colon K_*(SA) \to K_*(S\mathcal{T}_X)$ modulo the isomorphism $(S\varphi_0)_*$. We will show that it is also "the right inverse". To this end, we first "shift" the $*$-homomorphism $S\bar{\pi}_X\colon SA \to S\mathcal{T}_X$ along the $*$-homomorphism $S\varphi_0\colon SA \to D$ (see Lemma C.15).

Definition C.10. For each $n \in \mathbb{N}$, we set $Y_n = \overline{\operatorname{span}}(\bar{t}_X^n(X^{\otimes n})\mathcal{T}_X) \subset \mathcal{T}_X$, which is naturally a Hilbert $\mathcal{T}_X$-module. We denote by Y the direct sum of the Hilbert $\mathcal{T}_X$-modules $\{Y_n\}_{n=0}^\infty$.

Remark C.11. The Hilbert $\mathcal{T}_X$-module Y is isomorphic to the interior tensor product of the Hilbert A-module $\mathcal{F}(X)$ and the Hilbert $\mathcal{T}_X$-module $\mathcal{T}_X$ with the $*$-homomorphism $\bar{\pi}_X\colon A \to \mathcal{T}_X$.

The linear maps $\bar{t}_X^n\colon X^{\otimes n} \to Y_n$ extend a linear map $\bar{t}_X^\bullet\colon \mathcal{F}(X) \to Y$. By the definition, we get $Y = \overline{\operatorname{span}}(\bar{t}_X^\bullet(\mathcal{F}(X))\mathcal{T}_X)$. We also have $\langle \bar{t}_X^\bullet(\xi), \bar{t}_X^\bullet(\eta)\rangle_Y = \bar{\pi}_X\big(\langle \xi, \eta\rangle_{\mathcal{F}(X)}\big)$ for all $\xi, \eta \in \mathcal{F}(X)$.

Definition C.12. We define a $*$-homomorphism $\Phi\colon \mathcal{L}(\mathcal{F}(X)) \ni T \mapsto \Phi(T) \in \mathcal{L}(Y)$ by

$$\Phi(T)\big(\bar{t}^{\bullet}_X(\xi)x\big) = \bar{t}^{\bullet}_X(T(\xi))x \quad \text{ for } \xi \in \mathcal{F}(X) \text{ and } x \in \mathcal{T}_X.$$

It is not difficult to see that Φ is well-defined.

Lemma C.13. *We have* $\Phi\big(\mathcal{K}(\mathcal{F}(X))\big) \subset \mathcal{K}(Y)$.

Proof. This follows from the fact that $\Phi(\theta_{\xi,\eta}) = \theta_{\bar{t}^{\bullet}_X(\xi),\bar{t}^{\bullet}_X(\eta)}$ for $\xi, \eta \in \mathcal{F}(X)$, which is easily verified. $\square$

We define $\widetilde{D} = D\big(\mathcal{K}(Y), \mathcal{L}(Y)\big)$. By Lemma C.13, we can define a $*$-homomorphism $D\Phi\colon D \to \widetilde{D}$. Since we assume that X is non-degenerate, we have $Y_0 = \mathcal{T}_X$. Hence the natural isomorphism $\mathcal{T}_X \cong \mathcal{K}(Y_0) \subset \mathcal{K}(Y)$ gives us a $*$-homomorphism $\widetilde{\varphi}_0\colon \mathcal{T}_X \to \mathcal{K}(Y)$.

Lemma C.14. *The $*$-homomorphism $S\widetilde{\varphi}_0\colon S\mathcal{T}_X \to \widetilde{D}$ induces an isomorphism on the K-groups.*

Proof. Similar as the proof of Lemma C.8. $\square$

Lemma C.15. *We have the following commutative diagram;*

$$\begin{array}{ccc} SA & \xrightarrow{S\bar{\pi}_X} & S\mathcal{T}_X \\ \big\downarrow{\scriptstyle S\varphi_0} & & \big\downarrow{\scriptstyle S\widetilde{\varphi}_0} \\ D & \xrightarrow{D\Phi} & \widetilde{D}, \end{array}$$

Proof. Straightforward. $\square$

Proposition C.16. *The composition of $\rho\colon S\mathcal{T}_X \to D$ and $D\Phi\colon D \to \widetilde{D}$ induces an isomorphism on the K-groups.*

Proof. We set $\pi = \Phi \circ \varphi_\infty\colon A \to \mathcal{L}(Y)$. For each $s \in [0,1]$, we define a linear map $t_s\colon X \to \mathcal{L}(Y)$ by

$$t_s(\xi) = s\widetilde{\varphi}_0(\bar{t}_X(\xi)) + \sqrt{1-s^2}\Phi(\tau^1_0(\xi)) + \Phi(\tau^-_\infty(\xi))$$

It is routine to check that the pair (π, t_s) is a representation of X. Thus we get a $*$-homomorphism $\rho_s\colon \mathcal{T}_X \to \mathcal{L}(Y)$ such that $\rho_s \circ \bar{\pi}_X = \pi$ and $\rho_s \circ \bar{t}_X = t_s$ for each s. We have $\rho_0 = \Phi \circ \rho_+$ because $t_0 = \Phi \circ \tau_\infty$. We also have $\rho_1 = \widetilde{\varphi}_0 + \Phi \circ \rho_-$ because $t_1 = \widetilde{\varphi}_0 \circ \bar{t}_X + \Phi \circ \tau^-_\infty$ and $\pi = \widetilde{\varphi}_0 \circ \bar{\pi}_X + \Phi \circ \varphi^-_\infty$. For $\xi \in X$ and $s \in [0,1]$, we have $t_s(\xi) - \Phi(\tau^-_\infty(\xi)) \in \mathcal{K}(Y)$ because $\widetilde{\varphi}_0(\bar{t}_X(\xi)), \Phi(\tau^1_0(\xi)) \in \mathcal{K}(Y)$. Since we have $\pi(a) - \Phi(\varphi^-_\infty(a)) = \widetilde{\varphi}_0(\bar{\pi}_X(a)) \in \mathcal{K}(Y)$, we can prove $\rho_s(x) - \Phi(\rho_-(x)) \in \mathcal{K}(Y)$ for all $x \in \mathcal{T}_X$ and $s \in [0,1]$ in a similar way to the proof of Lemma C.7. Hence we can see that the composition of $D\Phi \circ \rho$ is homotopic to the $*$-homomorphism $S\mathcal{T}_X \to \widetilde{D}$ defined from the two $*$-homomorphisms $S\widetilde{\varphi}_0 + S\Phi \circ \rho_-$ and $S\Phi \circ \rho_-$. By Lemma C.5, we see that $D\Phi \circ \rho$ induces the same map as $S\widetilde{\varphi}_0$. Hence the proof completes by Lemma C.14. $\square$

Combining all the results above, we obtain that the composition of the map $\rho_*\colon K_*(S\mathcal{T}_X) \to K_*(D)$ and the isomorphism $(S\varphi_0)_*^{-1}\colon K_*(D) \to K_*(SA)$ gives the inverse of the map $(S\bar{\pi}_X)_*\colon K_*(SA) \to K_*(S\mathcal{T}_X)$. Hence we have shown that

$(\bar{\pi}_X)_*\colon K_*(A) \to K_*(\mathcal{T}_X)$ is an isomorphism when the C^*-correspondence X is non-degenerate. We will see that this is the case for general C^*-correspondences.

Let us take a C^*-correspondence X over a C^*-algebra A. We define

$$T = \overline{\operatorname{span}}(\bar{\pi}_X(A)\mathcal{T}_X\bar{\pi}_X(A))$$

which is the hereditary C^*-subalgebra of $\mathcal{T}_X$ generated by $\bar{\pi}_X(A)$. Since the ideal generated by $\bar{\pi}_X(A)$ is $\mathcal{T}_X$, Proposition B.5 shows that the inclusion $T \hookrightarrow \mathcal{T}_X$ induces an isomorphism on the K-groups. Hence to prove that the $*$-homomorphism $\bar{\pi}_X\colon A \to \mathcal{T}_X$ induces an isomorphism on the K-groups, it suffices to show that the $*$-homomorphism $\bar{\pi}_X\colon A \to T$ induces an isomorphism on the K-groups. This can be shown by applying the discussion above to the non-degenerate C^*-correspondence in the next lemma.

Lemma C.17. *Let us set $X' = \overline{\operatorname{span}}(\varphi_X(A)X)$ which is a non-degenerate C^*-correspondence over A. Then there exists an isomorphism $\rho\colon \mathcal{T}_{X'} \to T$ such that $\rho \circ \bar{\pi}_{X'} = \bar{\pi}_X$.*

Proof. Let us set $\pi = \bar{\pi}_X$ and define a linear map $t\colon X' \to \mathcal{T}_X$ as the restriction of $\bar{t}_X$ to X'. It is easy to see that the pair (π, t) is a representation of X'. Hence we have a $*$-homomorphism $\rho\colon \mathcal{T}_{X'} \to \mathcal{T}_X$. It is clear that the gauge action of $\mathcal{T}_X$ is a gauge action for the representation (π, t). It is also clear that $\{a \in A \mid \pi(a) \in \psi_t(\mathcal{K}(X'))\} = 0$. Hence ρ is injective by Theorem 6.2. Finally, it is not difficult to see that the image of ρ is T. $\square$

This completes the proof of Proposition 8.2.

References

[AEE] Abadie, B.; Eilers, S.; Exel, R. *Morita equivalence for crossed products by Hilbert C^*-bimodules.* Trans. Amer. Math. Soc. **350** (1998), no. 8, 3043–3054.

[BHRS] Bates, T.; Hong, J.; Raeburn, I.; Szymański, W. *The ideal structure of the C^*-algebras of infinite graphs.* Illinois J. Math. **46** (2002), no. 4, 1159–1176.

[B] Brown, L. G. *Stable isomorphism of hereditary subalgebras of C^*-algebras.* Pacific J. Math. **71** (1977), no. 2, 335–348.

[DLRZ] Doplicher, S.; Longo, R.; Roberts, J. E.; Zsido, L. *A remark on quantum group actions and nuclearity.* Rev. Math. Phys. **14** (2002), no. 7-8, 787–796.

[DS] Dykema, K.; Shlyakhtenko, D. *Exactness of Cuntz-Pimsner C^*-algebras.* Proc. Edinburgh Math. Soc. **44** (2001), 425-444.

[E] Exel, R. *A Fredholm operator approach to Morita equivalence.* K-Theory **7** (1993), no. 3, 285–308.

[FLR] Fowler, N. J.; Laca, M.; Raeburn, I. *The C^*-algebras of infinite graphs.* Proc. Amer. Math. Soc. **128** (2000), no. 8, 2319–2327.

[FMR] Fowler, N. J.; Muhly, P. S.; Raeburn, I. *Representations of Cuntz-Pimsner algebras.* Indiana Univ. Math. J. **52** (2003), no. 3, 569–605.

[HR] Higson, N.; Roe, J. *Analytic K-homology.* Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2000.

[KPW] Kajiwara, T.; Pinzari, C.; Watatani, Y. *Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules.* J. Funct. Anal. **159** (1998), no. 2, 295–322.

[Ka1] Katsura, T. *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras I, fundamental results.* Preprint 2002, math.OA/0207252.

[Ka2] Katsura, T. *A construction of C^*-algebras from C^*-correspondences.* To appear in Contemp. Math., Proceedings of the AMS conference "Advances in Quantum Dynamics".

[Ka3] Katsura, T. *Ideal structure of C^*-algebras associated with C^*-correspondences.* Preprint 2003, math.OA/0309294.

[Ki] Kirchberg, E. *On subalgebras of the CAR-algebra.* J. Funct. Anal. **129** (1995), no. 1, 35–63.

[KPR] Kumjian, A.; Pask, D.; Raeburn, I. *Cuntz-Krieger algebras of directed graphs.* Pacific J. Math. **184** (1998), no. 1, 161–174.

[KPRR] Kumjian, A.; Pask, D.; Raeburn, I.; Renault, J. *Graphs, groupoids, and Cuntz-Krieger algebras.* J. Funct. Anal. **144** (1997), no. 2, 505–541.

[L1] Lance, C. *On nuclear C^*-algebras.* J. Funct. Anal. **12** (1973), 157–176.

[L2] Lance, E. C. *Hilbert C^*-modules. A toolkit for operator algebraists.* London Mathematical Society Lecture Note Series, **210**. Cambridge University Press, Cambridge, 1995.

[MS] Muhly, P. S.; Solel, B. *Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes.* J. Funct. Anal. **158** (1998), no. 2, 389–457.

[P] Pimsner, M. V. *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z.* Free probability theory, 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

[RS] Rosenberg, J.; Schochet, C. *The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor.* Duke Math. J. **55** (1987), no. 2, 431–474.

[W] Wassermann, S. *Exact C^*-algebras and related topics.* Lecture Notes Series, 19. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994.

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo, 153-8914, JAPAN

Current address: Department of Mathematics, University of Oregon, Eugene, Oregon, 97403-1222, U.S.A.

E-mail address: katsu@ms.u-tokyo.ac.jp